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Distributed Control of Multiagent Systems under Heterogeneity

by

Selahattin Burak Sarsilmaz

A dissertation submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy in Mechanical Engineering
Department of Mechanical Engineering
College of Engineering
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Keywords: Cooperative Output Regulation, Internal Model, Synchronization Roles,
Global and Local Objectives

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Dedication

This work is dedicated to my mother Zerrin, who is my heroine.

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Abstract

The overarching objective of this work is to propose solutions to quite a few distributed control problems arising from networks of heterogeneous agents or the heterogeneous nature of multiagent systems. Each problem with its solutions is concisely summarized below.

We consider the cooperative output regulation problem of heterogeneous linear multiagent systems over fixed directed communication graphs. The purpose of this problem is to design a distributed control law such that the overall closed-loop stability is ensured and the tracking error of each agent converges to zero asymptotically for a class of reference inputs and disturbances generated by a so-called exosystem. We investigate the solvability of the problem with internal model-based distributed control laws, namely dynamic state feedback, dynamic output feedback with local measurement, and dynamic output feedback. The approach is twofold: First, the overall closed-loop stability (i.e., global property) is assumed and it is shown, under mild assumptions, that the problem is solved. Second, an agent-wise local sufficient condition is derived to guarantee the global property under standard assumptions.

Then, we update the definition of the linear cooperative output regulation problem to allow not only common output synchronization among agents but also an additional output synchronization among a proper subset of the agents for a distributed dynamic state feedback control law that does not exchange its state variables through a communication graph. Similar to the above-mentioned approach, its solvability is investigated by making use of the internal model design from the linear output regulation theory and a small-gain theorem for large-scale interconnected systems.

This dissertation also focuses on distributed control of linear multiagent systems with both global and local objectives over fixed directed communication graphs. While the global objective is achieving leaderless synchronization (i.e., consensus) or synchronization to a leader, local objectives for a subset of agents are tasks determined by agent-specific dynamical systems around the synchronization mapping of the global objective. Our main goal is to design a distributed control law such that each agent obeys the global objective when it is not assigned the local task and performs its own local objective otherwise. To

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this end, we construct reference models for all agents via two existing synchronization results, introduce two classes of distributed controllers, and formally define the considered problems. Then, they are solved by utilizing the converging-input converging-state property for a class of linear systems and the feedforward design approach from the linear output regulation theory.

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Chapter 1: Introduction

Over the last two decades, synchronization has been an immensely popular subject in the cooperative control literature because of its vast array of applications ranging from formation of mobile robots to surveillance with unmanned aerial vehicles (e.g., see [2–5] and references therein). From a control theory viewpoint, the fundamental problem in synchronization of coupled dynamical systems is the derivation of conditions that ensure state or output synchronization of the network of these agents [6, 7]. The main theoretical difficulty in this problem arises from the lack of a central authority; that is, the controller of each agent relies only on the information about the agent and its neighbors [8].

State synchronization in networks of *identical* systems (i.e., *homogeneous* multiagent systems) on directed graphs has been well studied. In particular, networks of single-integrator (respectively, double-integrator) agents are considered in [9–11] (respectively, [12, 13]). For general linear time-invariant dynamical systems, the authors of [6, 14–18] have proposed different distributed controllers and explored conditions to guarantee leaderless synchronization. Extensions to cooperative tracking (i.e., synchronization to a leader or leader-following consensus) problems have been further investigated in [17, 19, 20].

Although the aforementioned literature addresses the synchronization of homogeneous linear multiagent systems in detail, *heterogeneity* resulting from *nonidentical* system dynamics is inevitable for many real-world applications. For instance, coordination of autonomous vehicles of various sizes or kinds for environment mapping leads to synchronization problems in *heterogeneous* multivehicle systems. Instead of state synchronization, output synchronization among all agents is now expected since the state of every agent does not necessarily have the same physical interpretation. In fact, state dimensions can be different [8, 21, 22]. The authors of [22, 23] have derived necessary and sufficient conditions for output synchronization of networks of heterogeneous (in dynamics and dimension) linear time-invariant systems under directed graphs. In addition, *heterogeneity* can stem from *local objectives* of each agent, which are around the *global objective* of the multiagent system, even if individual systems are identical.

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This dissertation focuses on three facets of distributed control problems due to the heterogeneity mentioned in the preceding paragraph. The following sections briefly describe them and our contributions.

1.1 Cooperative Output Regulation of Heterogeneous Linear Multiagent Systems

The output regulation (i.e., servomechanism) problem has been one of the central research topics in control theory since the early 1970s. It mainly concentrates on controlling the output of an uncertain system to achieve asymptotic tracking with disturbance rejection for a class of reference inputs and disturbances generated by an exosystem, which is usually a known autonomous differential equation, while preserving the closed-loop stability [24, 25]. For general linear time-invariant systems, this problem was solved by [26–28], where the celebrated internal model principle of control theory was the significant outcome.

For large-scale systems such as power distribution networks and flexible manufacturing systems, decentralized design of decentralized controller (i.e., design of local controllers based on models of only the relevant parts of the system) is more efficient than its centralized counterpart in control theory. On the other hand, many problems solved by control theory are subproblems in decentralized control theory [29, 30]. The connections between these theories have been studied in [31, 32] for the decentralized output regulation problem of linear interconnected systems. It should be noted that each local controller is assumed to have an access to its reference input.

The cooperative output regulation (i.e., distributed output regulation) problem, which can be regarded as an extension of the conventional output regulation problem to multiagent systems, has attracted attention during the last decade (e.g., see [1, 33–48] and references therein). The objective of this problem is to design a distributed control law that enables overall closed-loop stability and output synchronization of all agents to the reference input in the presence of external disturbances. The reference input to be tracked and external disturbances to be rejected by the agents are generated by an exosystem as in the output regulation theory. However, the problem cannot be solved by a decentralized control scheme since the information of the exosystem is available to only a proper subset of all agents [34–37].

The cooperative output regulation problem has been studied in [36, 37] for the networks of almost identical linear agents on directed graphs. In contrast to the dynamics of the leader considered in [19, 20], the exosystem can differ from the unforced dynamics of the agents; hence, for example, a target with different dynamics and unmeasurable variables can be tracked by identical vehicles. Therefore, the problem formulation of cooperative output regulation can also be seen as an extension of the leader-following

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consensus problem when the exosystem and agents are viewed as the leader and followers, respectively [34–37].

The problem of cooperative output regulation of heterogeneous (in dynamics and dimension) linear time-invariant multiagent systems over general fixed (i.e., static) directed communication graphs have been recently investigated in [1, 34, 35, 38–40, 42–44] with numerous distributed controllers. Similar to the output regulation theory, distributed control approaches used to solve this problem can be classified into two approaches: Feedforward approach and internal model approach. The former is adopted by [34, 35, 38, 40, 42–44] and the feedforward gain of each agent relies on the solution of the regulator equations; hence, this approach is known to be not robust to parameter uncertainties. On the other hand, the latter employed by [1, 39] is robust with respect to small variations of the plant parameters. However, it cannot be applied when the transmission zero condition does not hold.

1.2 Linear Cooperative Output Regulation with Primary and Secondary Synchronization Roles

In Section 1.1, the papers on the linear cooperative output regulation problem of heterogeneous multiagent systems over general fixed directed communication graphs are classified according to the output regulation theory. The heterogeneity in dimensions of the regulated outputs is a desired feature if the agents are expected to track different dimensional reference signals. Now, we group the papers according to the regulated output dimensions of the agents. While they are allowed to be different in [34, 35, 40, 43], the authors of [1, 38, 39, 42, 44] assume that the dimensions are the same and even the output matrix of the exosystem for each agent is identical. Besides using the feedforward approach, the controllers in [34, 35, 40, 43] all have distributed observers of the exosystem that exchange information about their states over the communication graph. The proposed solutions in [1, 38, 39, 42], however, do not exchange information about states of the controllers. Instead, they utilize relative output information of neighboring agents.

When controllers are not allowed to exchange their state variables, we seek output synchronization among all agents in [1, 38, 39, 42], but is it the only possible synchronization in the network? To see this, consider a heterogeneous multiagent system that is not too "heterogeneous"; that is, some of the agents have similar dynamics. A simple example is a network of aerial and ground robots. Even if an exosystem generates a reference input consisting of positions in the x, y, and z directions, altitude synchronization of all robots cannot be achieved except trivial cases. On the contrary, positions in the x and y directions can be synchronized among all agents. Basically, the regulated output of each agent in this case is the *common*

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output of the exosystem and all the agents. In addition to the common output synchronization, which shall be called the *primary synchronization role* of the multiagent system, we can seek synchronization of all aerial robots in the z direction. We shall refer to the additional synchronization roles like the one in the example as the *secondary synchronization roles* of the multiagent system.

It should be noted that the problem formulation of [34, 35, 40, 43] inherently includes the primary and secondary synchronization roles of the multiagent systems. However, for distributed controllers that do not exchange their state variables through communication graphs, the linear cooperative output regulation problem with primary and secondary synchronization roles has not been studied yet.

1.3 Control of Linear Multiagent Systems with Global and Local Objectives

With the system-theoretic advancements in distributed control of multiagent systems, groups of agents are now able to utilize local information exchange for achieving a broad class of global objectives that range from consensus to containment. Despite all the developments in the multiagent system literature, the following fundamental question arises: *How do some of the agents forming the multiagent system perform their own local objectives, which are defined with respect to the global objective of the multiagent system, without deteriorating the overall multiagent system's global objective?*

In fact, this question has been recently raised in [49] by the authors and system-theoretically addressed in [49] by providing five different distributed controllers (i.e., protocols) with comparable advantages (see Tables I and II in [49]) for single-integrator agent dynamics when the global objective is leaderless consensus. In [50], these controllers are slightly modified to achieve the leader-follower consensus as a global objective. Furthermore, several experiments are conducted on a team of ground mobile robots with these protocols. This experimental evaluation has shown that the third and fifth distributed controllers in [49] and [50] outperform the other three for both leaderless and leader-follower consensus.

1.4 Contributions

In Chapter 2, we study the linear cooperative output regulation problem with the internal model approach. Thus, our study is relevant to the studies in [1, 39], where they have proposed a distributed dynamic state feedback control law incorporating a p-copy internal model of the system matrix of the exosystem. Moreover, [39] extends the results in [1] to an output feedback control under an output feedback stabilizability condition. Our contributions to this problem are as follows:

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- A considerable number of flaws in the results of [1, 39] is illustrated by counterexamples and fixed.
- The problem definition is slightly modified to show that the approach can also be useful when the exosystem is not known exactly in practical applications.
- We not only consider a distributed dynamic state feedback control law but also a distributed dynamic output feedback control law with local measurement output, where the output feedback stabilizability is not assumed, and a distributed dynamic output feedback control law, where agents have no access to their own states or outputs. For each control law, the solvability of the problem is investigated in two steps. First, a global property, which requires both the dynamics of every agent and the communication graph, is assumed and it is proved, under mild assumptions, that the problem is solved. Second, an agent-wise local sufficient condition, which paves the way for independent controller design for each agent, is provided to ensure the global property under standard assumptions.
- The proof technique utilized in the first step does not decompose the matrix equations that are crucial for the solvability of the problem, unlike the technique in [1, 39]. This helps us to weaken the assumptions of the first step.

It is worth noting that the proposed distributed controllers also solve the robust cooperative output regulation problem considered in [33, 37] for heterogeneous uncertain linear multiagent systems owing to the incorporation of a p-copy internal model of the system matrix of the exosystem into the controllers. Therefore, they are superior to the ones in [34, 35, 38, 40, 42–44], which use the feedforward approach, in terms of handling the plant uncertainties. Furthermore, with the proposed distributed control laws, each agent does not need the exchange of its controller's state variables. Instead, it makes use of relative output information between itself and its neighbors. Hence, the proposed controllers can be more practical than the controllers in [34, 35, 40, 43] when the agents are equipped with the sensors measuring the relative output.

In Chapter 3, the definition of the linear cooperative output regulation problem is updated to allow not only the primary synchronization role but also a secondary synchronization role for a distributed dynamic state feedback control law that does not rely on the exchange of its state variables. Similar to Chapter 2, the solvability of the updated problem is investigated by employing the internal model approach and a small-gain theorem.

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In Chapter 4, we study distributed control problems for high-order linear time-invariant multiagent systems with both global and local objectives over fixed directed communication graphs. While the considered global objective is to solve a typical leaderless synchronization or synchronization to a leader problem, local objectives for a subset of agents are tasks determined by agent-specific dynamical systems around the synchronization mapping of the global objective. Based on the existing synchronization results of [6] and [19], we construct (distributed) reference model, which achieves the global objective, for each agent. Building on the harmony of global and local objectives considered in [49] together with the third and fifth protocols of [49] and [50] for single-integrator agent dynamics, we introduce two classes of distributed controllers for high-order linear time-invariant agent dynamics and define the problems to be addressed. We then solve them by utilizing the converging-input converging-state property for a class of linear systems and the feedforward approach.

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Chapter 2: A Distributed Control Approach for Heterogeneous Linear Multiagent Systems[1]

This paper considers an internal model based distributed control approach to the cooperative output regulation problem of heterogeneous linear time-invariant multiagent systems over fixed directed communication graph topologies. First, a new definition of the linear cooperative output regulation problem is introduced in order to allow a broad class of functions to be tracked and rejected by a network of agents. Second, the solvability of this problem with three distributed control laws, namely dynamic state feedback, dynamic output feedback with local measurement, and dynamic output feedback, is investigated by first considering a global condition and then providing an agent-wise local sufficient condition under standard assumptions. Finally, two numerical examples are provided to illustrate the selected contributions of this paper.

2.1 Introduction

Heterogeneous multiagent systems formed by networks of agents having different dynamics and dimensions present a significantly broader class of multiagent systems than their heterogeneous and homogeneous counterparts that consist of networks of agents having different dynamics with the same dimension and identical dynamics, respectively. Therefore, analysis and synthesis of distributed control approaches for this class of multiagent systems that rely on local information exchange has been an attractive research topic in the systems and control field over the last decade.

In particular, the cooperative output regulation problem of heterogeneous (in dynamics and dimension) linear time-invariant multiagent systems, where the output of all agents synchronize to the output of the leader, over general fixed directed communication graph topologies have been recently investigated in [1, 34, 35, 38, 39, 42, 43]. This problem can be regarded as the generalization of the linear output regulation problem given in, for example, [25] to multiagent systems. As a consequence, distributed control approaches to this regulation problem can be classified into two categories:

[1]This chapter is previously published in [51]. Permission is included in Appendix E.

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- The first category is predicated on feedforward design methodology, where the authors of [34, 35, 38, 42, 43] present contributions. In the presence of plant uncertainties, however, this methodology is known to be not robust since the feedforward gain of each agent relies on the solution of the regulator equations.

- The second category is predicated on internal model principle, where the authors of [1, 39] present contributions. While this methodology is robust with respect to small variations of the plant parameters as compared to feedforward design methodology, it cannot be applied when the transmission zero condition does not hold.

The common denominator of these papers is that an exosystem, which has an unforced linear time-invariant dynamics, generates both a reference trajectory and external disturbances to be tracked and rejected by networks of agents. Specifically, the system matrix of the exosystem is explicitly used by controllers of all agents in [34, 35, 38, 42] and a proper subset of agents in [43]; or each agent incorporates a p-copy internal model of this matrix in its controller [1, 39].

2.1.1 Contributions

Considering applications of the distributed control approaches in [1, 34, 35, 38, 39, 42, 43], it can be a challenge to precisely know the system matrix of the exosystem, even the dynamical structure of the exosystem; especially, when an external leader interacts with the network of agents or a control designer simply injects optimized trajectory commands to the network based on, for example, an online path planning algorithm. In order to guarantee ultimately bounded tracking error in such cases, a new, generalized definition for the cooperative output regulation problem is needed.

This paper focuses on heterogeneous (in dynamics and dimension) linear time-invariant multiagent systems over general fixed directed communication graph topologies. First, we present the generalized definition for the linear cooperative output regulation problem. Second, we investigate the solvability of this problem for internal model based distributed dynamic state feedback, output feedback with local measurement, and output feedback control laws. To this end, we not only consider global conditions but also provide agent-wise local sufficient conditions under standard assumptions. Considering large-scale applications of multiagent systems, the agent-wise local sufficient conditions are primarily important for independent controller design of each agent (i.e., without depending on the dynamics of other agents).

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The system-theoretical approach presented in this paper[2] is relevant to the studies in [1, 39], where they also focus on the linear cooperative output regulation problem with an internal model based distributed dynamic state feedback control law. Specifically, [39] extends the approach in [1] to an output feedback control under an output feedback stabilizability condition. In addition to the generalized definition of the linear cooperative output regulation problem, the contribution of this paper differs from the studies in [1, 39] based on the following points:

- First, we note that the theoretical contribution of this paper covers not only the dynamic state feedback problem but also the dynamic output feedback problem with local measurement as well as the dynamic output feedback problem. Unlike the results presented in [39], this paper does not assume the output feedback stabilizability for the dynamic output feedback problem with local measurement. With regard to the dynamic output feedback problem, the results of this paper does not require agents to access their own states or outputs.

- To prove the existence of a unique solution to the matrix equations that are crucial for the solvability of the problem, Section III in [1] (Theorem 4 in [39]) decomposes these matrix equations, which consist of the overall dynamics of the multiagent system, into matrix equations, which deal with the dynamics of each agent separately. In contrast, we do not decompose these matrix equations; see the sixth paragraph of Appendix A for the advantage. In particular, Lemma 2.4.3 of this paper, which is also applicable to dynamic output feedback cases, guarantees that these matrix equations have a unique solution without requiring their decompositions.

- A considerable number of gaps in the related results of [1, 39] is illustrated by counterexamples in Appendices A and B and fixed in Appendices A and B as well as in Section 2.4.1.

2.1.2 Organization

The rest of the paper is organized as follows. Section 2.2 presents the notation and the essential mathematical preliminaries. Section 2.3 formulates the linear cooperative output regulation problem considered in this paper. The solvability of this problem is investigated in Section 2.4 and two illustrative numerical examples are presented in Section 2.5. Finally, Section 2.6 concludes the paper.

[2] Although they are not completely related, [52, 53] may be regarded as preliminary works of this paper.

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2.2 Mathematical Preliminaries

A standard notation is used in this paper. Specifically, $\mathbb{R}$, $\mathbb{R}^n$, and $\mathbb{R}^{n\times m}$ respectively denote the sets of all real numbers, $n\times 1$ real column vectors, and $n\times m$ real matrices[3]; $\mathbf{1}_n$ and I_n respectively denote the $n\times 1$ vector of all ones and the $n\times n$ identity matrix; and "$\triangleq$" denotes equality by definition. We write $(\cdot)^{\mathrm{T}}$ for the transpose and $\|\cdot\|_2$ for the induced two norm of a matrix; $\sigma(\cdot)$ for the spectrum[4] and $\rho(\cdot)$ for the spectral radius of a square matrix; $(\cdot)^{-1}$ for the inverse of a nonsingular matrix; and $\otimes$ for the Kronecker product. We also write $A\leq B$ for $A\in\mathbb{R}^{n\times m}$, $B\in\mathbb{R}^{n\times m}$ if entries $a_{ij}\leq b_{ij}$ for all ordered pairs (i,j). Finally, $\mathrm{diag}(A_1,\ldots,A_n)$ is a block-diagonal matrix with matrix entries $A_1,\ldots,A_n$ on its diagonal.

We now concisely state the graph theoretical notation used in this paper, which is based on [5]. In particular, consider a fixed (i.e., time-invariant) directed graph $\mathcal{G}=(\mathcal{V},\mathcal{E})$, where $\mathcal{V}=\{v_1,\ldots,v_N\}$ is a nonempty finite set of N nodes and $\mathcal{E}\subset\mathcal{V}\times\mathcal{V}$ is a set of edges. Each node in $\mathcal{V}$ corresponds to a follower agent. There is an edge rooted at node v_j and ended at v_i (i.e., $(v_j,v_i)\in\mathcal{E}$) if and only if v_i receives information from v_j. $\mathcal{A}=[a_{ij}]\in\mathbb{R}^{N\times N}$ denotes the adjacency matrix, which describes the graph structure; that is, $a_{ij}>0\Leftrightarrow(v_j,v_i)\in\mathcal{E}$ and $a_{ij}=0$ otherwise. Repeated edges and self loops are not allowed; that is, $a_{ii}=0$, $\forall i\in\mathcal{N}$ with $\mathcal{N}=\{1,\ldots,N\}$. The set of neighbors of node v_i is denoted as $N_i=\{j\mid(v_j,v_i)\in\mathcal{E}\}$. In-degree matrix is defined by $\mathcal{D}=\mathrm{diag}(d_1,\ldots,d_N)$ with $d_i=\sum_{j\in N_i}a_{ij}$. A directed path from node v_i to node v_j is a sequence of successive edges in the form $\{(v_i,v_p),(v_p,v_q),\ldots,(v_r,v_j)\}$. If $v_i=v_j$, then the directed path is called a loop. A directed graph is said to have a spanning tree if there is a root node such that it has directed paths to all other nodes in the graph. A fixed augmented directed graph is defined as $\bar{\mathcal{G}}=(\bar{\mathcal{V}},\bar{\mathcal{E}})$, where $\bar{\mathcal{V}}=\{v_0,v_1,\ldots,v_N\}$ is the set of $N+1$ nodes, including leader node v_0 and all nodes in $\mathcal{V}$, and $\bar{\mathcal{E}}=\mathcal{E}\cup\mathcal{E}'$ is the set of edges with $\mathcal{E}'$ consisting of some edges in the form of (v_0,v_i), $i\in\mathcal{N}$.

The concept of internal model introduced next slightly modifies Definition 1.22 and Remark 1.24 in [25].

Definition 2.2.1 *Given any square matrix A_0, a triple of matrices (M_1,M_2,M_3) is said to incorporate a p-copy internal model of the matrix A_0 if*

$$M_1=T\begin{bmatrix}S_1 & S_2\\ 0 & G_1\end{bmatrix}T^{-1},\ M_2=T\begin{bmatrix}S_3\\ G_2\end{bmatrix},\ M_3=T\begin{bmatrix}S_4\\ 0\end{bmatrix}, \tag{2.1}$$

[3] In this paper, all real matrices are defined over the field of complex numbers.
[4] We follow Definition 4.4.4 in [54].

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or

$$M_1 = G_1,\ M_2 = G_2,\ M_3 = 0, \tag{2.2}$$

where S_l, $l = 1,2,3,4$, is any matrix with an appropriate dimension, T is any nonsingular matrix with an appropriate dimension, the zero matrix in M_3 has as many rows as those of G_1, and

$$G_1 = \mathrm{diag}(\beta_1, \ldots, \beta_p), \quad G_2 = \mathrm{diag}(\sigma_1, \ldots, \sigma_p),$$

where for $l = 1, \ldots, p$, $\beta_l \in \mathbb{R}^{s_l \times s_l}$ and $\sigma_l \in R^{s_l}$ satisfy the following conditions:

a) The pair (β_l, σ_l) is controllable.

b) The minimal polynomial of A_0 is equal to the characteristic polynomial of β_l.

2.3 Problem Formulation

Consider a system of N (follower) agents with heterogeneous linear time-invariant dynamics subject to external disturbances over a fixed directed communication graph topology $\mathcal{G}$. The dynamics of agent $i \in \mathcal{N}$ is given by

$$\begin{aligned} \dot{x}_i(t) &= A_i x_i(t) + B_i u_i(t) + \delta_i(t), \quad x_i(0) = x_{i0}, \quad t \geq 0, \\ y_i(t) &= C_i x_i(t) + D_i u_i(t), \end{aligned}$$

with state $x_i(t) \in \mathbb{R}^{n_i}$, input $u_i(t) \in \mathbb{R}^{m_i}$, output $y_i(t) \in \mathbb{R}^p$, and external disturbance $\delta_i(t) = E_{\delta_i}\delta(t) \in \mathbb{R}^{n_i}$, where $\delta(t) \in \mathbb{R}^{q_\delta}$ is a solution to the unknown disturbance dynamics with an initial condition. In addition, the reference trajectory to be tracked is denoted by $y_0(t) = R_{\mathrm{r}} r_0(t) \in \mathbb{R}^p$, where $r_0(t) \in \mathbb{R}^{q_{\mathrm{r}}}$ is a solution to the unknown leader dynamics with an initial condition.

Let $\omega(t) \triangleq [r_0^{\mathrm{T}}(t), \delta^{\mathrm{T}}(t)]^{\mathrm{T}} \in \mathbb{R}^q$ be the solution of the unknown exosystem, where $q = q_{\mathrm{r}} + q_\delta$. Instead of assuming that the exosystem has an unforced linear time-invariant dynamics with a known system matrix (e.g., see [1, 34, 39]), we consider that the exosystem has an unknown dynamics. From this perspective, the exosystem can represent any (e.g., linear or nonlinear) dynamics provided that its solution is unique and satisfies the conditions given later in Assumptions 2.3.1 and 2.3.2.

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Define $E_i \triangleq [0\ E_{\delta_i}]$ and $R \triangleq [R_{\mathrm{r}}\ 0]$. Furthermore, let $e_i(t) \triangleq y_i(t) - y_0(t)$ be the tracking error. We can then write the dynamics of each agent and its tracking error as

$$\dot{x}_i(t) = A_i x_i(t) + B_i u_i(t) + E_i \omega(t), \quad x_i(0) = x_{i0}, \quad t \geq 0, \tag{2.3}$$

$$e_i(t) = C_i x_i(t) + D_i u_i(t) - R\omega(t). \tag{2.4}$$

In this paper, the tracking error $e_i(t)$ is available to a nonempty proper subset of agents[5]. In particular, if node v_i observes the leader node v_0, then there exists an edge (v_0, v_i) with weighting gain $k_i > 0$; otherwise $k_i = 0$. Each agent has also access to the relative output error; that is, $y_i(t) - y_j(t)$ for all $j \in N_i$. Similar to [39], the local virtual tracking error can be defined as

$$e_{\mathrm{v}i}(t) \triangleq \frac{1}{d_i + k_i}\Big[\sum_{j \in N_i} a_{ij}\big(y_i(t) - y_j(t)\big) + k_i\big(y_i(t) - y_0(t)\big)\Big]. \tag{2.5}$$

Now, we define three classes of distributed control laws based on additional available information to each agent:

1) *Dynamic State Feedback.* If each agent has full access to its own state $x_i(t)$, then the dynamic state feedback control law is given by

$$u_i(t) = K_{1i} x_i(t) + K_{2i} z_i(t), \tag{2.6}$$

$$\dot{z}_i(t) = G_{1i} z_i(t) + G_{2i} e_{\mathrm{v}i}(t), \quad z_i(0) = z_{i0}, \quad t \geq 0, \tag{2.7}$$

where $z_i(t) \in \mathbb{R}^{n_{z_{1i}}}$ is the controller state and the quadruple $(K_{1i}, K_{2i}, G_{1i}, G_{2i})$ is specified in Section 2.4.1.

2) *Dynamic Output Feedback with Local Measurement.* If each agent has local measurement output $y_{\mathrm{m}i}(t) \in \mathbb{R}^{p_i}$ of the form

$$y_{\mathrm{m}i}(t) = C_{\mathrm{m}i} x_i(t) + D_{\mathrm{m}i} u_i(t), \tag{2.8}$$

then the dynamic output feedback control law with local measurement is given by

[5] If all agents observe the leader, decentralized controllers can be designed for each agent even though the distributed controllers proposed here are still applicable.

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$$u_i(t) = \bar{K}_i z_i(t), \tag{2.9}$$

$$\dot{z}_i(t) = M_{1i} z_i(t) + M_{2i} e_{\mathrm{v}i}(t) + M_{3i} y_{\mathrm{m}i}(t), \quad z_i(0) = z_{i0}, \quad t \geq 0, \tag{2.10}$$

where $z_i(t) \in \mathbb{R}^{n_{z_{2i}}}$ is the controller state and the quadruple $(\bar{K}_i, M_{1i}, M_{2i}, M_{3i})$ is specified in Section 2.4.2.

3) *Dynamic Output Feedback.* If each agent does not have additional information; that is, the local virtual tracking error $e_{\mathrm{v}i}(t)$ is the only available information to it, then the dynamic output feedback control law is given by

$$u_i(t) = \bar{K}_i z_i(t), \tag{2.11}$$

$$\dot{z}_i(t) = M_{1i} z_i(t) + M_{2i} e_{\mathrm{v}i}(t), \quad z_i(0) = z_{i0}, \quad t \geq 0, \tag{2.12}$$

where $z_i(t) \in \mathbb{R}^{n_{z_{2i}}}$ is the controller state and the triple $(\bar{K}_i, M_{1i}, M_{2i})$ is specified in Section 2.4.3.

We now introduce the first and the second assumptions before defining the problem.

Assumption 2.3.1 *$A_0 \in \mathbb{R}^{q \times q}$ has no eigenvalues with negative real parts.*

Assumption 2.3.2 *There exists $\kappa > 0$ such that*

$$\|A_0 \omega(t) - \dot{\omega}(t)\|_2 \leq \kappa < \infty, \quad \forall t \geq 0,$$

where $\dot{\omega}(t)$ is a piecewise continuous function[6] *of t.*

Assumption 2.3.1 is standard in linear output regulation theory (e.g., see Remark 1.3 in [25]). Assumption 2.3.2 is required to show the ultimate boundedness of the tracking error and it automatically holds if the exosystem has an unforced linear time-invariant dynamics with the system matrix A_0. Note that these assumptions do not imply the exact knowledge of the exosystem. We refer to Remarks 2.4.2 and 2.4.3 for further discussions and Section 2.5 for illustrative examples on this point.

Based on the definition of the linear cooperative output regulation problem in [1, 34], the problem considered in this paper is defined as follows.

[6] We follow the definition given in page 650 of [55].

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Definition 2.3.1 *Given the system in* (2.3) *and* (2.4) *together with the exosystem, which satisfies Assumptions 2.3.1 and 2.3.2, and the fixed augmented directed graph* $\bar{\mathcal{G}}$*, find a distributed control law of the form* (2.6) *and* (2.7)*, or* (2.9) *and* (2.10)*, or* (2.11) *and* (2.12) *such that:*

a) The resulting closed-loop system matrix is Hurwitz.

b) The tracking error $e_i(t)$ *is ultimately bounded with ultimate bound* b *for all initial conditions of the closed-loop system and for all* $i \in \mathcal{N}$*; that is, there exists* $b > 0$ *and for each initial condition of the closed-loop system, there is* $T \geq 0$ *such that* $\|e_i(t)\|_2 \leq b$, $\forall t \geq T$, $\forall i \in \mathcal{N}$.

c) If $\lim_{t\to\infty} A_0\omega(t) - \dot{\omega}(t) = 0$*, then for all initial conditions of the closed-loop system* $\lim_{t\to\infty} e_i(t) = 0$, $\forall i \in \mathcal{N}$.

This paper makes the following additional assumptions to solve this problem.

Assumption 2.3.3 *The fixed augmented directed graph* $\bar{\mathcal{G}}$ *has a spanning tree with the root node being the leader node.*

Assumption 2.3.4 *The pair* (A_i, B_i) *is stabilizable for all* $i \in \mathcal{N}$.

Assumption 2.3.5 *For all* $\lambda \in \sigma(A_0)$,

$$\operatorname{rank}\begin{bmatrix} A_i - \lambda I_{n_i} & B_i \\ C_i & D_i \end{bmatrix} = n_i + p, \quad \forall i \in \mathcal{N}.$$

Assumption 2.3.6 *As in* (2.2)*, the triple* $(G_{1i}, G_{2i}, 0)$ *incorporates a p-copy internal model of* A_0 *for all* $i \in \mathcal{N}$.

Assumption 2.3.7 *The pair* $(A_i, C_{\mathrm{m}i})$ *is detectable for all* $i \in \mathcal{N}$.

Assumption 2.3.8 *The pair* (A_i, C_i) *is detectable for all* $i \in \mathcal{N}$.

Assumption 2.3.3 is natural to solve the stated problem (e.g., see Remark 3.2 in [5]). Similar to Assumption 2.3.1, Assumptions 2.3.4-2.3.8 are standard in linear output regulation theory (e.g., see Chapter 1 of [25]). We use Assumptions 2.3.1-2.3.6 for dynamic state feedback. To utilize some results from dynamic state feedback in the absence of full state information, each agent requires the estimation of its own state. For this purpose, Assumption 2.3.7 and Assumption 2.3.8 are included for dynamic output feedback with local measurement and dynamic output feedback, respectively.

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2.4 Solvability of the Problem

For the three different distributed control laws introduced in Section 2.3, this section investigates the solvability of the problem given in Definition 2.3.1. Specifically, the approach in this section is twofold. First, the property $a)$ of Definition 2.3.1 is assumed and it is shown, under mild conditions, that the properties $b)$ and $c)$ of Definition 2.3.1 are satisfied. Second, an agent-wise local sufficient condition (i.e., distributed criterion) is provided for the property $a)$ of Definition 2.3.1 (i.e., the stability of the closed-loop system matrix) under standard assumptions.

Before studying the solvability of the problem for each distributed control law, we now present some definitions that are used throughout this section to express the closed-loop systems in compact forms, some results related to the communication graph topology, and a key lemma about the solvability of matrix equations, which play a crucial role on the solvability of the problem.

Define the following matrices: $\Phi \triangleq \mathrm{diag}(\Phi_1,\ldots,\Phi_N)$, $\Phi = A,B,C,D,E$; $\Phi_{\mathrm{m}} \triangleq \mathrm{diag}(\Phi_{\mathrm{m}1},\ldots,\Phi_{\mathrm{m}N})$, $\Phi = C,D$; $K_l \triangleq \mathrm{diag}(K_{l1},\ldots,K_{lN})$, $l = 1,2$; $A_{0\mathrm{a}} \triangleq I_N \otimes A_0$, and $R_{\mathrm{a}} \triangleq I_N \otimes R$. Further, let $x(t) \triangleq [x_1^{\mathrm{T}}(t),\ldots,x_N^{\mathrm{T}}(t)]^{\mathrm{T}} \in \mathbb{R}^{\bar{n}}$, where $\bar{n} = \sum_{i=1}^{N} n_i$; $e(t) \triangleq [e_1^{\mathrm{T}}(t),\ldots,e_N^{\mathrm{T}}(t)]^{\mathrm{T}} \in \mathbb{R}^{Np}$, $e_{\mathrm{v}}(t) \triangleq [e_{\mathrm{v}1}^{\mathrm{T}}(t),\ldots,e_{\mathrm{v}N}^{\mathrm{T}}(t)]^{\mathrm{T}} \in \mathbb{R}^{Np}$, and $\omega_{\mathrm{a}}(t) \triangleq \mathbf{1}_N \otimes \omega(t) \in \mathbb{R}^{Nq}$.

Observing $y_i(t) - y_j(t) = e_i(t) - e_j(t)$ and recalling $d_i = \sum_{j\in N_i} a_{ij}$, (2.5) can be equivalently written as

$$e_{\mathrm{v}i}(t) \quad = \quad e_i(t) - \frac{1}{d_i + k_i}\sum_{j\in N_i} a_{ij} e_j(t). \tag{2.13}$$

Let $\mathcal{F} \triangleq \mathrm{diag}\left(\frac{1}{d_1+k_1},\ldots,\frac{1}{d_N+k_N}\right)$ and $\mathcal{W} \triangleq (I_N - \mathcal{F}\mathcal{A}) \otimes I_p$. Here, it should be noted that $d_i + k_i > 0$, $\forall i \in \mathcal{N}$ by Assumption 2.3.3; hence, $\mathcal{F}$ is well-defined. From (2.13), we have

$$e_{\mathrm{v}}(t) \quad = \quad \mathcal{W} e(t). \tag{2.14}$$

Similar to Lemma 3.3 in [5], we next present the following lemma for $I_N - \mathcal{F}\mathcal{A}$.

Lemma 2.4.1 *Under Assumption 2.3.3, $I_N - \mathcal{F}\mathcal{A}$ is nonsingular. In addition, all its eigenvalues have positive real parts.*

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Proof. Under Assumption 2.3.3, $I_N - \mathcal{F}\mathcal{A}$ satisfies the conditions of the theorem in [56]. Thus, it is nonsingular. Since the singularity is eliminated, all the eigenvalues of $I_N - \mathcal{F}\mathcal{A}$ have positive real parts by the Gershgorin circle theorem (e.g., see Fact 4.10.17 in [54]). ■

Remark 2.4.1 *Since $I_N - \mathcal{F}\mathcal{A}$ is nonsingular under Assumption 2.3.3, so is $\mathcal{W}$ by Proposition 7.1.7 in [54]. Then, it is clear from* (2.14) *that $e_i(t)$ is bounded for all $i \in \mathcal{N}$ if and only if $e_{\mathrm{v}i}(t)$ is bounded for all $i \in \mathcal{N}$; $\lim_{t\to\infty} e_i(t) = 0$, $\forall i \in \mathcal{N}$ if and only if $\lim_{t\to\infty} e_{\mathrm{v}i}(t) = 0$, $\forall i \in \mathcal{N}$.*

We now investigate the spectral radius of $\mathcal{F}\mathcal{A}$.

Lemma 2.4.2 *Under Assumption 2.3.3, $\rho(\mathcal{F}\mathcal{A}) < 1$.*

Proof. By Lemma 2.4.1, all the eigenvalues of $I_N - \mathcal{F}\mathcal{A}$ have positive real parts under Assumption 2.3.3. This directly implies from Fact 6.2.1.4 in [57] that the leading principal minors of $I_N - \mathcal{F}\mathcal{A}$ are all positive as $I_N - \mathcal{F}\mathcal{A}$ is a square matrix whose off-diagonal elements are all nonpositive. Since $\mathcal{F}\mathcal{A}$ is a nonnegative square matrix and the leading principal minors of $I_N - \mathcal{F}\mathcal{A}$ are all positive, $\rho(\mathcal{F}\mathcal{A}) < 1$ from Lemma 6.2.1.8 in [57]. ■

Finally, we introduce the key lemma that extends the field of application of Lemma 1.27 in [25] to heterogeneous (in dynamics and dimension) linear time-invariant multiagent systems over general fixed directed communication graph topologies.

Lemma 2.4.3[7] *Let Assumptions 2.3.1 and 2.3.3 hold. Suppose the triple (M_1, M_2, M_3) incorporates an Np-copy internal model of $A_{0\mathrm{a}}$. If*

$$A_{\mathrm{c}} \triangleq \begin{bmatrix} \hat{A} & \hat{B} \\ M_2\mathcal{W}\hat{C} + M_3\hat{C}_{\mathrm{m}} & M_1 + M_2\mathcal{W}\hat{D} + M_3\hat{D}_{\mathrm{m}} \end{bmatrix}$$

is Hurwitz, where $\hat{A}$, $\hat{B}$, $\hat{C}$, $\hat{C}_{\mathrm{m}}$, $\hat{D}$, and $\hat{D}_{\mathrm{m}}$ are any matrices with appropriate dimensions, then the matrix equations

$$XA_{0\mathrm{a}} = \hat{A}X + \hat{B}Z + \hat{E}, \tag{2.15}$$

$$ZA_{0\mathrm{a}} = M_1Z + M_2\mathcal{W}(\hat{C}X + \hat{D}Z + \hat{F}) + M_3(\hat{C}_{\mathrm{m}}X + \hat{D}_{\mathrm{m}}Z), \tag{2.16}$$

[7] To investigate the solvability of a matrix equation that is obtained for a different problem setting with the distributed dynamic state feedback control law, the authors of [58] utilized the same logic in the proof of Lemma 2.4.3 (see Section 3.1 in [58]).

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have unique solutions X and Z for any matrices $\hat{E}$ and $\hat{F}$ of appropriate dimensions. Furthermore, X and Z satisfy

$$0 = \hat{C}X + \hat{D}Z + \hat{F}. \tag{2.17}$$

In other words, the conclusion is that the matrix equations

$$X_{\mathrm{c}}A_{0\mathrm{a}} = A_{\mathrm{c}}X_{\mathrm{c}} + B_{\mathrm{c}}, \tag{2.18}$$

$$0 = C_{\mathrm{c}}X_{\mathrm{c}} + D_{\mathrm{c}}, \tag{2.19}$$

have a unique solution X_{c}, where

$$X_{\mathrm{c}} = \begin{bmatrix} X \\ Z \end{bmatrix}, \; B_{\mathrm{c}} = \begin{bmatrix} \hat{E} \\ M_2\mathcal{W}\hat{F} \end{bmatrix}, \; C_{\mathrm{c}} = \begin{bmatrix} \hat{C} & \hat{D} \end{bmatrix}, \; D_{\mathrm{c}} = \hat{F}.$$

Proof. Note that (2.15) and (2.16) (respectively, (2.17)) can be equivalently written as (2.18) (respectively, (2.19)). Note also that $\sigma(A_{0\mathrm{a}}) = \sigma(A_0)$. Since Assumption 2.3.1 holds and A_{c} is Hurwitz, $A_{0\mathrm{a}}$ and A_{c} have no eigenvalues in common. Thus, the Sylvester equation in (2.18) has a unique solution $X_{\mathrm{c}} = [X^{\mathrm{T}}\ Z^{\mathrm{T}}]^{\mathrm{T}}$ by the first part of Proposition A.2 in [25]. In addition, we show that X and Z also satisfy (2.17). To this end, let $\bar{\gamma} \triangleq \hat{C}X + \hat{D}Z + \hat{F}$. Since the triple (M_1, M_2, M_3) incorporates an Np-copy internal model of $A_{0\mathrm{a}}$, it has the form given by (2.1) or (2.2). If it takes the form (2.1), let $[\hat{\theta}^{\mathrm{T}}\ \bar{\theta}^{\mathrm{T}}]^{\mathrm{T}} \triangleq T^{-1}Z$, where $\bar{\theta}$ has as many rows as those of G_1. Premultiplying (2.16) by T^{-1} and using the foregoing definitions, we obtain

$$\bar{\theta}A_{0\mathrm{a}} = G_1\bar{\theta} + G_2\mathcal{W}\bar{\gamma}. \tag{2.20}$$

Note that if the triple (M_1, M_2, M_3) takes the form (2.2), (2.16) already satisfies (2.20), where $\bar{\theta} = Z$. Let $\gamma \triangleq \mathcal{W}\bar{\gamma}$; then, (2.20) is in the form of (1.74) in [25]. Hence, $\gamma = 0$ by the proof of Lemma 1.27 in [25]. We know from Remark 2.4.1 that $\mathcal{W}$ is nonsingular under Assumption 2.3.3. As a consequence, $\gamma = 0$ implies $\bar{\gamma} = 0$. This completes the proof of this lemma. ■

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2.4.1 Dynamic State Feedback

Let $z(t) \triangleq [z_1^{\mathrm{T}}(t), \ldots, z_N^{\mathrm{T}}(t)]^{\mathrm{T}} \in \mathbb{R}^{\bar{n}_{z_1}}$, where $\bar{n}_{z_1} = \sum_{i=1}^{N} n_{z_{1i}}$, and $G_l \triangleq \mathrm{diag}(G_{l1}, \ldots, G_{lN})$, $l = 1, 2$. Inserting (2.6) into (2.3) and (2.4), and using the above definitions, (2.3), (2.7), and (2.4) can be compactly written as

$$\dot{x}(t) = (A + BK_1)x(t) + BK_2 z(t) + E\omega_{\mathrm{a}}(t), \quad x(0) = x_0, \quad t \geq 0, \tag{2.21}$$

$$\dot{z}(t) = G_1 z(t) + G_2 e_{\mathrm{v}}(t), \quad z(0) = z_0, \quad t \geq 0, \tag{2.22}$$

$$e(t) = (C + DK_1)x(t) + DK_2 z(t) - R_{\mathrm{a}}\omega_{\mathrm{a}}(t). \tag{2.23}$$

Next, insert (2.23) into (2.14) and replace the obtained expression with the one in (2.22). Define $x_{\mathrm{g}}(t) \triangleq [x^{\mathrm{T}}(t), z^{\mathrm{T}}(t)]^{\mathrm{T}} \in \mathbb{R}^{\bar{n}+\bar{n}_{z_1}}$. Then, the closed-loop system of (2.3)-(2.7) becomes

$$\dot{x}_{\mathrm{g}}(t) = A_{\mathrm{g}} x_{\mathrm{g}}(t) + B_{\mathrm{g}}\omega_{\mathrm{a}}(t), \quad x_{\mathrm{g}}(0) = x_{\mathrm{g}0}, \quad t \geq 0, \tag{2.24}$$

$$e(t) = C_{\mathrm{g}} x_{\mathrm{g}}(t) + D_{\mathrm{g}}\omega_{\mathrm{a}}(t), \tag{2.25}$$

where

$$A_{\mathrm{g}} = \begin{bmatrix} A + BK_1 & BK_2 \\ G_2\mathcal{W}(C + DK_1) & G_1 + G_2\mathcal{W}DK_2 \end{bmatrix}, \; B_{\mathrm{g}} = \begin{bmatrix} E \\ -G_2\mathcal{W}R_{\mathrm{a}} \end{bmatrix},$$

$$C_{\mathrm{g}} = \begin{bmatrix} C + DK_1 & DK_2 \end{bmatrix}, \; D_{\mathrm{g}} = -R_{\mathrm{a}}.$$

Theorem 2.4.1 *Let Assumptions 2.3.1-2.3.3 and 2.3.6 hold. If A_{g} is Hurwitz, then the distributed dynamic state feedback control given by* (2.6) *and* (2.7) *solves the problem in Definition 2.3.1.*

Proof. By the definition of $A_{0\mathrm{a}}$, the minimal polynomials for $A_{0\mathrm{a}}$ and A_0 are the same. Thus, the triple $(G_1, G_2, 0)$ incorporates an Np-copy internal model of $A_{0\mathrm{a}}$ under Assumption 2.3.6. Let $(M_1, M_2, M_3) \triangleq (G_1, G_2, 0)$. Let also $\hat{A} \triangleq A + BK_1$, $\hat{B} \triangleq BK_2$, $\hat{C} \triangleq C + DK_1$, $\hat{C}_{\mathrm{m}} \triangleq 0$, $\hat{D} \triangleq DK_2$, $\hat{D}_{\mathrm{m}} \triangleq 0$, $\hat{E} \triangleq E$, and $\hat{F} \triangleq -R_{\mathrm{a}}$. Then, the quadruple $(A_{\mathrm{g}}, B_{\mathrm{g}}, C_{\mathrm{g}}, D_{\mathrm{g}})$ takes the form of $(A_{\mathrm{c}}, B_{\mathrm{c}}, C_{\mathrm{c}}, D_{\mathrm{c}})$ in Lemma 2.4.3. In addition, A_{g} is Hurwitz and Assumptions 2.3.1 and 2.3.3 hold. Hence, Lemma 2.4.3 is applicable and it implies that the

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matrix equations

$$X_{\mathrm{g}}A_{0\mathrm{a}} = A_{\mathrm{g}}X_{\mathrm{g}} + B_{\mathrm{g}}, \tag{2.26}$$

$$0 = C_{\mathrm{g}}X_{\mathrm{g}} + D_{\mathrm{g}}, \tag{2.27}$$

have a unique solution X_{g}. We also refer to Appendix A for additional discussions on the solvability of (2.26) and (2.27).

Under Assumption 2.3.2, $\|A_{0\mathrm{a}}\omega_{\mathrm{a}}(t) - \dot{\omega}_{\mathrm{a}}(t)\|_2 \leq \sqrt{N}\kappa$, $\forall t \geq 0$ since $\|A_{0\mathrm{a}}\omega_{\mathrm{a}}(t) - \dot{\omega}_{\mathrm{a}}(t)\|_2^2 = N\|A_0\omega(t) - \dot{\omega}(t)\|_2^2$. Let $\bar{x}_{\mathrm{g}}(t) \triangleq x_{\mathrm{g}}(t) - X_{\mathrm{g}}\omega_{\mathrm{a}}(t)$. Then, using the definition of $\bar{x}_{\mathrm{g}}(t)$ and (2.26) and (2.27), we can rewrite (2.24) and (2.25) as

$$\dot{\bar{x}}_{\mathrm{g}}(t) = A_{\mathrm{g}}\bar{x}_{\mathrm{g}}(t) + X_{\mathrm{g}}(A_{0\mathrm{a}}\omega_{\mathrm{a}}(t) - \dot{\omega}_{\mathrm{a}}(t)), \quad \bar{x}_{\mathrm{g}}(0) = \bar{x}_{\mathrm{g}0}, \quad t \geq 0, \tag{2.28}$$

$$e(t) = C_{\mathrm{g}}\bar{x}_{\mathrm{g}}(t). \tag{2.29}$$

Now, the solution of (2.28) can be written as

$$\bar{x}_{\mathrm{g}}(t) = e^{A_{\mathrm{g}}t}\bar{x}_{\mathrm{g}0} + \int_0^t e^{A_{\mathrm{g}}(t-\tau)}X_{\mathrm{g}}(A_{0\mathrm{a}}\omega_{\mathrm{a}}(\tau) - \dot{\omega}_{\mathrm{a}}(\tau))\mathrm{d}\tau.$$

Since A_{g} is Hurwitz, there exist $c > 0$ and $\alpha > 0$ such that $\|e^{A_{\mathrm{g}}t}\|_2 \leq ce^{-\alpha t}$, $\forall t \geq 0$ (e.g., see Lecture 8.3 in [59]). Owing to this bound and the bound on $\|A_{0\mathrm{a}}\omega_{\mathrm{a}}(t) - \dot{\omega}_{\mathrm{a}}(t)\|_2$, we have the following inequality

$$\|\bar{x}_{\mathrm{g}}(t)\|_2 \leq ce^{-\alpha t}\|\bar{x}_{\mathrm{g}0}\|_2 + \frac{c\|X_{\mathrm{g}}\|_2}{\alpha}\sqrt{N}\kappa, \quad \forall t \geq 0.$$

Using the fact $\|e_i(t)\|_2 \leq \|e(t)\|_2$, $\forall i \in \mathcal{N}$ and observing $\|e(t)\|_2 \leq \|C_{\mathrm{g}}\|_2\|\bar{x}_{\mathrm{g}}(t)\|_2$ from (2.29), we arrive

$$\|e_i(t)\|_2 \leq ce^{-\alpha t}\|C_{\mathrm{g}}\|_2\|\bar{x}_{\mathrm{g}0}\|_2 + b', \quad \forall t \geq 0, \quad \forall i \in \mathcal{N},$$

where $b' = c\|C_{\mathrm{g}}\|_2\|X_{\mathrm{g}}\|_2\sqrt{N}\kappa\alpha^{-1}$. For a given $\varepsilon > 0$, we have either $c\|C_{\mathrm{g}}\|_2\|\bar{x}_{\mathrm{g}0}\|_2 > \varepsilon$ or $c\|C_{\mathrm{g}}\|_2\|\bar{x}_{\mathrm{g}0}\|_2 \leq \varepsilon$. In the former case, it can be readily shown that $ce^{-\alpha t}\|C_{\mathrm{g}}\|_2\|\bar{x}_{\mathrm{g}0}\|_2 \leq \varepsilon$, $\forall t \geq T$ with $T = \alpha^{-1}\ln\left(\frac{c\|C_{\mathrm{g}}\|_2\|\bar{x}_{\mathrm{g}0}\|_2}{\varepsilon}\right) > 0$. In the latter case, the foregoing inequality trivially holds for all $t \geq 0$. Thus, $e_i(t)$ is ultimately bounded with the ultimate bound $b \triangleq b' + \varepsilon$ for all $\bar{x}_{\mathrm{g}0}$, which is also true for all $x_{\mathrm{g}0}$, and for all $i \in \mathcal{N}$.

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If $\lim_{t\to\infty} A_0\omega(t) - \dot{\omega}(t) = 0$, then $\lim_{t\to\infty} A_{0\mathrm{a}}\omega_{\mathrm{a}}(t) - \dot{\omega}_{\mathrm{a}}(t) = 0$. Since A_{g} is Hurwitz and the system in (2.28) is linear time-invariant when $A_{0\mathrm{a}}\omega_{\mathrm{a}}(t) - \dot{\omega}_{\mathrm{a}}(t)$ is viewed as an input to the system, (2.28) is input-to-state stable with respect to this piecewise continuous input (e.g., see Chapter 4.9 in [55]). Thus, $\lim_{t\to\infty} A_{0\mathrm{a}}\omega_{\mathrm{a}}(t) - \dot{\omega}_{\mathrm{a}}(t) = 0$ implies $\lim_{t\to\infty} \bar{x}_{\mathrm{g}}(t) = 0$ for all $\bar{x}_{\mathrm{g}0}$ (e.g., see Exercise 4.58 in [55]). Finally, it follows from (2.29) that for all $x_{\mathrm{g}0}$ $\lim_{t\to\infty} e_i(t) = 0,\ \forall i \in \mathcal{N}$. ■

Remark 2.4.2 *The ultimate bound b of the tracking error for each agent is associated with the bound κ in Assumption 2.3.2. Specifically, as κ decreases (respectively, increases), b decreases (respectively, increases). To elucidate the role of Assumptions 2.3.1 and 2.3.2 in practice, we consider the following possible scenarios:*

a) When the piecewise continuity and boundedness of $\dot{\omega}(t)$ are the only information that is available to a control designer, the triple $(0, I_p, 0)$ incorporating a p-copy internal model of $A_0 = 0$ is quite natural; hence, (2.7) becomes a distributed integrator. Moreover, X_{g} in b can be explicitly expressed in terms of A_{g} and B_{g}; that is, $X_{\mathrm{g}} = -A_{\mathrm{g}}^{-1}B_{\mathrm{g}}$ by (2.26).

b) When the piecewise continuity and boundedness of $\dot{\omega}(t)$, the boundedness of $\omega(t)$, and some frequencies in $\omega(t)$ are available to a control designer, the triple $(G_{1i}, G_{2i}, 0)$ incorporating a p-copy internal model of A_0, which includes these frequencies and zero eigenvalues, is an alternative to the pure distributed integrator.

Remark 2.4.3 *As it is shown in Theorem 2.4.1, asymptotic synchronization is achieved when $\lim_{t\to\infty} A_0\omega(t) - \dot{\omega}(t) = 0$. We now provide sufficient conditions to check this condition as follows[8]. If one of the following conditions holds*

a) $\dot{\omega}(t) = A_0\omega(t), \quad \omega(0) = \omega_0, \quad t \geq 0$;

b) $\lim_{t\to\infty} e^{A_0 t}\omega_0 - \omega(t) = 0$, where $\omega_0 = \omega(0)$, and $A_0 e^{A_0 t}\omega_0 - \dot{\omega}(t)$ is uniformly continuous on $[0, \infty)$,

then $\lim_{t\to\infty} A_0\omega(t) - \dot{\omega}(t) = 0$. Note that a) clearly implies b). From Barbalat's lemma given by Lemma 8.2 in [60], b) implies that $\lim_{t\to\infty} A_0 e^{A_0 t}\omega_0 - \dot{\omega}(t) = 0$. Thus, $\lim_{t\to\infty} A_0\omega(t) - \dot{\omega}(t) = A_0 \lim_{t\to\infty} \omega(t) - e^{A_0 t}\omega_0 + \lim_{t\to\infty} A_0 e^{A_0 t}\omega_0 - \dot{\omega}(t) = 0$. In general, asymptotic synchronization results in the literature (e.g., see [1, 34, 39]) are obtained under the condition a). It is clear that this paper covers all class of functions generated under the condition a).

[8] If $A_0 = 0$, one should read $\lim_{t\to\infty} \dot{\omega}(t) = 0$ in place of $\lim_{t\to\infty} A_0\omega(t) - \dot{\omega}(t) = 0$; hence, $\omega(t) \equiv \omega^\star$ ($\omega^\star$ is finite) in place of a), and $\lim_{t\to\infty} \omega(t) = \omega^\star$ and $\dot{\omega}(t)$ is uniformly continuous on $[0, \infty)$ in place of b).

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To obtain an agent-wise local sufficient condition assuring the property a) of Definition 2.3.1 under some standard assumptions, let $\xi_i(t) \triangleq [x_i^{\mathrm{T}}(t), z_i^{\mathrm{T}}(t)]^{\mathrm{T}} \in \mathbb{R}^{n_i+n_{z_{1i}}}$, $\mu_i(t) \triangleq \frac{1}{d_i+k_i}\sum_{j\in N_i} a_{ij}e_j(t)$,

$$\bar{A}_i \triangleq \begin{bmatrix} A_i & 0 \\ G_{2i}C_i & G_{1i} \end{bmatrix}, \ \bar{B}_i \triangleq \begin{bmatrix} B_i \\ G_{2i}D_i \end{bmatrix}, \ B_{\mathrm{f}i} \triangleq \begin{bmatrix} 0 \\ -G_{2i} \end{bmatrix},$$

and $\bar{C}_i \triangleq [C_i \ 0]$. Furthermore, consider (2.3), (2.7), (2.13), and (2.4) when $\omega(t) \equiv 0$. We now have

$$\dot{\xi}_i(t) = \bar{A}_i\xi_i(t) + \bar{B}_i u_i(t) + B_{\mathrm{f}i}\mu_i(t), \quad \xi_i(0) = \xi_{i0}, \quad t \geq 0, \tag{2.30}$$

$$e_i(t) = \bar{C}_i\xi_i(t) + D_i u_i(t). \tag{2.31}$$

Next, define the matrices

$$A_{\mathrm{f}i} \triangleq \begin{bmatrix} A_i + B_iK_{1i} & B_iK_{2i} \\ G_{2i}(C_i + D_iK_{1i}) & G_{1i} + G_{2i}D_iK_{2i} \end{bmatrix},$$

$$C_{\mathrm{f}i} \triangleq \begin{bmatrix} C_i + D_iK_{1i} & D_iK_{2i} \end{bmatrix}.$$

Using (2.6), (2.30) and (2.31) can be written as

$$\dot{\xi}_i(t) = A_{\mathrm{f}i}\xi_i(t) + B_{\mathrm{f}i}\mu_i(t), \quad \xi_i(0) = \xi_{i0}, \quad t \geq 0, \tag{2.32}$$

$$e_i(t) = C_{\mathrm{f}i}\xi_i(t). \tag{2.33}$$

Let, in addition, $\Psi_{\mathrm{f}} \triangleq \mathrm{diag}(\Psi_{\mathrm{f}1}, \ldots, \Psi_{\mathrm{f}N})$, $\Psi = A, B, C$ and $\xi(t) \triangleq [\xi_1^{\mathrm{T}}(t), \ldots, \xi_N^{\mathrm{T}}(t)]^{\mathrm{T}}$. Then, (2.32) and (2.33) can be put into the compact form given by

$$\dot{\xi}(t) = A_{\mathrm{f}}\xi(t) + B_{\mathrm{f}}(\mathcal{F}\mathcal{A} \otimes I_p)\tilde{w}(t), \quad \xi(0) = \xi_0, \quad t \geq 0, \tag{2.34}$$

$$\tilde{z}(t) = C_{\mathrm{f}}\xi(t), \tag{2.35}$$

where $e(t) = \tilde{w}(t) = \tilde{z}(t)$. Observe that the system in (2.34) and (2.35) takes the form of (12) in [1]. Therefore, one may think of resorting Theorem 2 in [1] at first sight. However, the statement of Theorem 2 in [1] is not correct as it is written; we refer to Appendix B for a counterexample.

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This paragraph uses the notation and the terminology from [1]. Readers are referred to (12), Theorem 1, Theorem 2, and Lemma 8 in [1]. It should be noted that Theorem 2 relies on Theorem 1 and this theorem is derived by means of Theorem 11.8 and Lemma 11.2 in [61]. According to the mentioned results and Chapter 5.3, which is devoted to the notion of internal stability for the system of interest, in [61], it is clear that the following condition should be added to the hypotheses of Theorem 1: *Let the realization of $T(s)$ given by (12) be stabilizable and detectable.* With this modification, not only the theoretical gap in Theorem 1 but also the one in Theorem 2 is filled. However, a simple point in the proof of Theorem 2 still needs to be clarified. The spectral radius of $\tilde{T}(j\omega)$ in the proof of Theorem 2 is upper bounded by applying Lemma 8. Since Lemma 8 is applied, we infer that $\mathrm{diag}(\|T_1(j\omega)\|,\ldots,\|T_N(j\omega)\|)$ is regarded as a positive definite diagonal matrix, but its proof is not given. The foregoing diagonal matrix is necessarily positive semidefinite; hence, we only question[9] whether $T_i(s)=0$ for some i. Instead of investigating the corresponding realizations, we extend Lemma 8 to positive semidefinite diagonal matrices as follows.

Lemma 2.4.4 *Let $Q\in\mathbb{R}^{n\times n}$ be a nonnegative matrix. If $\Lambda\in\mathbb{R}^{n\times n}$ is a positive semidefinite diagonal matrix, then $\rho(\Lambda Q)\le\rho(\Lambda)\rho(Q)$.*

Proof. Let $\Lambda\triangleq\mathrm{diag}(\lambda_1,\ldots,\lambda_n)$ be positive semidefinite. If $\Lambda=0$, the inequality holds trivially. We therefore assume that there exists a $\lambda_i>0$ for some i; hence, $\rho(\Lambda)>0$. Let $\bar{\Lambda}\triangleq\mathrm{diag}(\bar{\lambda}_1,\ldots,\bar{\lambda}_n)$, where $\bar{\lambda}_i=\rho(\Lambda)$ if $\lambda_i=0$, $\bar{\lambda}_i=\lambda_i$ otherwise. By construction, $\Lambda\le\bar{\Lambda}$, $\rho(\Lambda)=\rho(\bar{\Lambda})$, and $\bar{\Lambda}$ is a positive definite diagonal matrix. Since $\Lambda\le\bar{\Lambda}$ and Q is nonnegative, $\Lambda Q\le\bar{\Lambda}Q$. By the corollary in page 27 of [62], $\rho(\Lambda Q)\le\rho(\bar{\Lambda}Q)$. Applying Lemma 8 in [1] to $\bar{\Lambda}Q$, we also have $\rho(\bar{\Lambda}Q)\le\rho(\bar{\Lambda})\rho(Q)$. Since $\rho(\Lambda)=\rho(\bar{\Lambda})$, we establish the desired inequality. ■

It is well known that the system in (2.34) and (2.35) is stabilizable and detectable if A_{f} is Hurwitz. Thus, the new condition is satisfied if $A_{\mathrm{f}i}$ is Hurwitz for all $i\in\mathcal{N}$.

Remark 2.4.4 *Assumptions 2.3.4-2.3.6 ensure the stabilizability of the pair $(\bar{A}_i,\bar{B}_i)$ for all $i\in\mathcal{N}$ by Lemma 1.26 in [25]. Therefore, K_{1i} and K_{2i} can always be chosen such that $A_{\mathrm{f}i}$ is Hurwitz for all $i\in\mathcal{N}$.*

Let $g_{\mathrm{f}i}(s)\triangleq C_{\mathrm{f}i}(sI-A_{\mathrm{f}i})^{-1}B_{\mathrm{f}i}$. We now state the following theorem for the dynamic state feedback case.

[9] Considering Kalman decomposition (e.g., see Theorem 16.3 in [59]), one can easily construct a linear time-invariant system with Hurwitz system matrix, nonzero input and output matrices, and zero direct feedthrough matrix such that its transfer matrix is zero.

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Theorem 2.4.2 *Let Assumption 2.3.3 hold and $A_{\mathrm{f}i}$ be Hurwitz for all $i \in \mathcal{N}$. If*

$$\|g_{\mathrm{f}i}\|_\infty \rho(\mathcal{F}\mathcal{A}) < 1, \quad \forall i \in \mathcal{N}, \tag{2.36}$$

where $\|g_{\mathrm{f}i}\|_\infty$ is the H_∞ norm of $g_{\mathrm{f}i}(s)$, then A_{g} is Hurwitz.

Proof. It follows from Theorem 2 in [1] and the above discussion. ■

Remark 2.4.5 *The inequality given by* (2.36) *is an agent-wise local sufficient condition; that is, it paves the way for independent controller design for each agent. For the connection between this condition and an algebraic Riccati equation (respectively, linear matrix inequality), we refer to Lemma 9 in [1] (respectively, Theorem 6 in [39]). Moreover, we know from Lemma 2.4.2 that $\rho(\mathcal{F}\mathcal{A}) < 1$ under Assumption 2.3.3. Therefore, we can restate Theorem 2.4.2 by replacing* (2.36) *with $\|g_{\mathrm{f}i}\|_\infty \leq 1$, $\forall i \in \mathcal{N}$. In this statement, although the condition becomes more conservative, it is not only agent-wise local but also graph-wise local except Assumption 2.3.3. Finally, it should be noted that if the graph $\mathcal{G}$ considered in Theorem 2.4.2 contains no loop (i.e., acyclic), then the nodes in $\mathcal{G}$ can be relabelled such that $i > j$ when $(v_j, v_i) \in \mathcal{E}$. Thus, $\mathcal{A}$ is similar to a lower triangular matrix with zero diagonal entries, so is $\mathcal{F}\mathcal{A}$. This implies that $\rho(\mathcal{F}\mathcal{A}) = 0$; hence, Theorem 2.4.2 does not require the condition given by* (2.36) *anymore. In terms of being agent-wise and graph-wise local, this special case is consistent with the result in [33].*

2.4.2 Dynamic Output Feedback with Local Measurement

Let $z_i(t) \triangleq [\hat{x}_i^{\mathrm{T}}(t), \bar{z}_i^{\mathrm{T}}(t)]^{\mathrm{T}} \in \mathbb{R}^{n_{z_{2i}}}$, where $\hat{x}_i(t)$ is the estimate of the state $x_i(t)$, $\bar{K}_i \triangleq [K_{1i}\ K_{2i}]$, and (2.9) have the form given by

$$u_i(t) \quad = \quad K_{1i}\hat{x}_i(t) + K_{2i}\bar{z}_i(t). \tag{2.37}$$

To estimate the state $x_i(t)$, the following local Luenberger observer is employed

$$\dot{\hat{x}}_i(t) \quad = \quad A_i\hat{x}_i(t) + B_i u_i(t) + H_i\big(y_{\mathrm{m}i}(t) - C_{\mathrm{m}i}\hat{x}_i(t) - D_{\mathrm{m}i}u_i(t)\big), \quad \hat{x}_i(0) = \hat{x}_{i0}, \quad t \geq 0, \tag{2.38}$$

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where H_i is the observer gain matrix. Using (2.37), we can write (2.38) as

$$\dot{\hat{x}}_i(t) = \big(A_i + B_iK_{1i} - H_i(C_{\mathrm{m}i} + D_{\mathrm{m}i}K_{1i})\big)\hat{x}_i(t) + H_iy_{\mathrm{m}i}(t) + (B_i - H_iD_{\mathrm{m}i})K_{2i}\bar{z}_i(t),$$
$$\hat{x}_i(0) = \hat{x}_{i0}, \quad t \geq 0. \quad (2.39)$$

Let also $\bar{z}_i(t)$ evolve according to the dynamics given by

$$\dot{\bar{z}}_i(t) = G_{1i}\bar{z}_i(t) + G_{2i}e_{\mathrm{v}i}(t), \quad \bar{z}_i(0) = \bar{z}_{i0}, \quad t \geq 0. \quad (2.40)$$

By (2.39) and (2.40), one can define the triple (M_{1i}, M_{2i}, M_{3i}) in (2.10) as

$$M_{1i} \triangleq \begin{bmatrix} A_i + B_iK_{1i} - H_i(C_{\mathrm{m}i} + D_{\mathrm{m}i}K_{1i}) & (B_i - H_iD_{\mathrm{m}i})K_{2i} \\ 0 & G_{1i} \end{bmatrix},$$
$$M_{2i} \triangleq \begin{bmatrix} 0 \\ G_{2i} \end{bmatrix}, \; M_{3i} \triangleq \begin{bmatrix} H_i \\ 0 \end{bmatrix}. \quad (2.41)$$

Using (2.8) and (2.37), (2.38) can be rewritten as

$$\dot{\hat{x}}_i(t) = H_iC_{\mathrm{m}i}x_i(t) + (A_i + B_iK_{1i} - H_iC_{\mathrm{m}i})\hat{x}_i(t) + B_iK_{2i}\bar{z}_i(t), \quad \hat{x}_i(0) = \hat{x}_{i0}, \quad t \geq 0. \quad (2.42)$$

Next, define $\hat{x}(t) \triangleq [\hat{x}_1^{\mathrm{T}}(t), \ldots, \hat{x}_N^{\mathrm{T}}(t)]^{\mathrm{T}}$, $\bar{z}(t) \triangleq [\bar{z}_1^{\mathrm{T}}(t), \ldots, \bar{z}_N^{\mathrm{T}}(t)]^{\mathrm{T}}$, and $H \triangleq \mathrm{diag}(H_1, \ldots, H_N)$. Inserting (2.37) into (2.3) and (2.4), using (2.42), (2.40), and the above definitions, (2.3), (2.10), and (2.4) can be compactly written as

$$\dot{x}(t) = Ax(t) + BK_1\hat{x}(t) + BK_2\bar{z}(t) + E\omega_{\mathrm{a}}(t), \qquad x(0) = x_0, \quad t \geq 0, \quad (2.43)$$
$$\dot{\hat{x}}(t) = HC_{\mathrm{m}}x(t) + (A + BK_1 - HC_{\mathrm{m}})\hat{x}(t) + BK_2\bar{z}(t), \qquad \hat{x}(0) = \hat{x}_0, \quad t \geq 0, \quad (2.44)$$
$$\dot{\bar{z}}(t) = G_1\bar{z}(t) + G_2e_{\mathrm{v}}(t), \qquad \bar{z}(0) = \bar{z}_0, \quad t \geq 0, \quad (2.45)$$
$$e(t) = Cx(t) + DK_1\hat{x}(t) + DK_2\bar{z}(t) - R_{\mathrm{a}}\omega_{\mathrm{a}}(t). \quad (2.46)$$

Now, insert (2.46) into (2.14) and replace the obtained expression with the one in (2.45). Let $\eta(t) \triangleq [x^{\mathrm{T}}(t), \hat{x}^{\mathrm{T}}(t), \bar{z}^{\mathrm{T}}(t)]^{\mathrm{T}} \in \mathbb{R}^{\bar{n}+\bar{n}_{z_2}}$, where $\bar{n}_{z_2} = \sum_{i=1}^{N} n_{z_{2i}}$. Then, the closed-loop system of (2.3)-(2.5) and (2.8)-

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(2.10) can be represented as

$$\dot{\eta}(t) = A_\eta \eta(t) + B_\eta \omega_{\mathrm{a}}(t), \quad \eta(0) = \eta_0, \quad t \geq 0, \tag{2.47}$$

$$e(t) = C_\eta \eta(t) + D_\eta \omega_{\mathrm{a}}(t), \tag{2.48}$$

where

$$A_\eta = \begin{bmatrix} A & BK_1 & BK_2 \\ HC_{\mathrm{m}} & A + BK_1 - HC_{\mathrm{m}} & BK_2 \\ G_2\mathcal{W}C & G_2\mathcal{W}DK_1 & G_1 + G_2\mathcal{W}DK_2 \end{bmatrix},$$

$$B_\eta = \begin{bmatrix} E \\ 0 \\ -G_2\mathcal{W}R_{\mathrm{a}} \end{bmatrix}, \; C_\eta = \begin{bmatrix} C & DK_1 & DK_2 \end{bmatrix}, \; D_\eta = -R_{\mathrm{a}}.$$

For the following result, we define $A_{\mathrm{H}i} \triangleq A_i - H_i C_{\mathrm{m}i}$ and $A_{\mathrm{H}} \triangleq A - HC_{\mathrm{m}}$. By Assumption 2.3.7, H_i can always be chosen such that $A_{\mathrm{H}i}$ is Hurwitz for all $i \in \mathcal{N}$.

Theorem 2.4.3 *Let Assumptions 2.3.1-2.3.3 and 2.3.6 hold. If A_{g} is Hurwitz and $A_{\mathrm{H}i}$ is Hurwitz for all $i \in \mathcal{N}$, then the distributed dynamic output feedback control with local measurement given by* (2.9) *and* (2.10) *solves the problem in Definition 2.3.1.*

Proof. Let $K \triangleq [K_1 \; K_2]$, $\hat{A} \triangleq A$, $\hat{B} \triangleq BK$, $\hat{C} \triangleq C$, $\hat{C}_{\mathrm{m}} \triangleq C_{\mathrm{m}}$, $\hat{D} \triangleq DK$, $\hat{D}_{\mathrm{m}} \triangleq D_{\mathrm{m}}K$, $\hat{E} \triangleq E$, $\hat{F} \triangleq -R_{\mathrm{a}}$,

$$M_1 \triangleq \begin{bmatrix} A + BK_1 - H(C_{\mathrm{m}} + D_{\mathrm{m}}K_1) & (B - HD_{\mathrm{m}})K_2 \\ 0 & G_1 \end{bmatrix},$$

$$M_2 \triangleq \begin{bmatrix} 0 \\ G_2 \end{bmatrix}, \; M_3 \triangleq \begin{bmatrix} H \\ 0 \end{bmatrix}. \tag{2.49}$$

Now, observe that the quadruple $(A_\eta, B_\eta, C_\eta, D_\eta)$ takes the form of $(A_{\mathrm{c}}, B_{\mathrm{c}}, C_{\mathrm{c}}, D_{\mathrm{c}})$ in Lemma 2.4.3. Recall from the proof of Theorem 2.4.1 that the triple $(G_1, G_2, 0)$ incorporates an Np-copy internal model of $A_{0\mathrm{a}}$ under Assumption 2.3.6. This clearly implies that the triple (M_1, M_2, M_3) also incorporates an Np-copy internal model of $A_{0\mathrm{a}}$. It is given that Assumptions 2.3.1 and 2.3.3 hold. In order to apply Lemma 2.4.3, we need to show that A_η is Hurwitz under the conditions that A_{g} is Hurwitz and $A_{\mathrm{H}i}$ is Hurwitz for all $i \in \mathcal{N}$.

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To this end, the following elementary row and column operations are performed on A_η. First, subtract row 1 from row 2 and add column 2 to column 1. Second, interchange rows 2 and 3, and interchange columns 2 and 3. Thus, we obtain the matrix given by

$$\bar{A}_\eta \triangleq \begin{bmatrix} A+BK_1 & BK_2 & BK_1 \\ G_2\mathcal{W}(C+DK_1) & G_1+G_2\mathcal{W}DK_2 & G_2\mathcal{W}DK_1 \\ 0 & 0 & A_{\mathrm{H}} \end{bmatrix}.$$

Considering the performed elementary row and column operations, one can verify that A_η is similar to $\bar{A}_\eta$; hence, they have the same eigenvalues. Since $\bar{A}_\eta$ is upper block triangular, $\sigma(\bar{A}_\eta) = \sigma(A_{\mathrm{g}}) \cup \sigma(A_{\mathrm{H}})$. Note that A_{H} is Hurwitz as $A_{\mathrm{H}i}$ is Hurwitz for all $i \in \mathcal{N}$. It is also given that A_{g} is Hurwitz. Thus, A_η is Hurwitz. Then, the matrix equations

$$\begin{aligned} X_\eta A_{0\mathrm{a}} &= A_\eta X_\eta + B_\eta, \\ 0 &= C_\eta X_\eta + D_\eta, \end{aligned}$$

have a unique solution X_η by Lemma 2.4.3.

Following similar steps to those in the proof of Theorem 2.4.1, it can be shown under Assumption 2.3.2 that $e_i(t)$ is ultimately bounded with an ultimate bound for all η_0 and for all $i \in \mathcal{N}$. If, in addition, $\lim_{t\to\infty} A_0\omega(t) - \dot{\omega}(t) = 0$, then for all η_0 $\lim_{t\to\infty} e_i(t) = 0$, $\forall i \in \mathcal{N}$. ■

Remark 2.4.6 *Since the condition on $A_{\mathrm{H}i}$ is both agent-wise and graph-wise local, obtaining an agent-wise local sufficient condition that ensures the property a) of Definition 2.3.1 boils down to finding an agent-wise local sufficient condition, under standard assumptions, for the stability of A_{g}, which is already given in Theorem 2.4.2.*

2.4.3 Dynamic Output Feedback

Define $z_i(t)$, $\bar{K}_i$, and $u_i(t)$ as in Section 2.4.2; that is, (2.11) has the form (2.37). Since $e_{\mathrm{v}i}(t)$ is the only available information to each agent, the following distributed observer is considered instead of (2.39)

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to estimate the state $x_i(t)$

$$\dot{\hat{x}}_i(t) = \big(A_i + B_i K_{1i} - L_i(C_i + D_i K_{1i})\big)\hat{x}_i(t) + L_i e_{\mathrm{v}i}(t) + (B_i - L_i D_i)K_{2i}\bar{z}_i(t),$$
$$\hat{x}_i(0) = \hat{x}_{i0}, \quad t \geq 0, \tag{2.50}$$

where L_i is the observer gain matrix. Let $\bar{z}_i(t)$ satisfy the dynamics in (2.40). We can now define the pair (M_{1i}, M_{2i}) in (2.12) by replacing the triple $(H_i, C_{\mathrm{m}i}, D_{\mathrm{m}i})$ in M_{1i} (respectively, the zero matrix in M_{2i}) given by (2.41) with (L_i, C_i, D_i) (respectively, L_i).

Define $\hat{x}(t)$ and $\bar{z}(t)$ as in the previous subsection and $L \triangleq \mathrm{diag}(L_1, \ldots, L_N)$. Inserting (2.37) into (2.3) and (2.4), using (2.50), (2.40), and the above definitions, (2.3), (2.12), and (2.4) can be expressed by (2.43),

$$\dot{\hat{x}}(t) = \big(A + BK_1 - L(C + DK_1)\big)\hat{x}(t) + (B - LD)K_2\bar{z}(t) + Le_{\mathrm{v}}(t), \quad \hat{x}(0) = \hat{x}_0, \quad t \geq 0, \tag{2.51}$$

(2.45), and (2.46). Next, insert (2.46) into (2.14) and replace the obtained expression not only with the one in (2.45) but also with the one in (2.51). In addition, define $\eta(t)$ as in Section 2.4.2. Then, the closed-loop system of (2.3)-(2.5), (2.11), and (2.12) can be expressed by (2.47) and (2.48) if the second row of A_η is replaced with

$$\begin{bmatrix} L\mathcal{W}C & A + BK_1 - L(C + DK_1 - \mathcal{W}DK_1) & (B - LD + L\mathcal{W}D)K_2 \end{bmatrix}$$

and the second row of B_η is replaced with $-L\mathcal{W}R_{\mathrm{a}}$.

Theorem 2.4.4 *Let Assumptions 2.3.1-2.3.3 and 2.3.6 hold. If the resulting A_η is Hurwitz, then the distributed dynamic output feedback control given by* (2.11) *and* (2.12) *solves the problem in Definition 2.3.1.*

Proof. Define K, $\hat{A}$, $\hat{B}$, $\hat{C}$, $\hat{D}$, $\hat{E}$, and $\hat{F}$ as in the proof of Theorem 2.4.3. Let $\hat{C}_{\mathrm{m}} \triangleq 0$, $\hat{D}_{\mathrm{m}} \triangleq 0$, and $M_3 \triangleq 0$. Define also the pair (M_1, M_2) by replacing the triple $(H, C_{\mathrm{m}}, D_{\mathrm{m}})$ in M_1 (respectively, the zero matrix in M_2) given by (2.49) with (L, C, D) (respectively, L). Then, observe that the resulting quadruple $(A_\eta, B_\eta, C_\eta, D_\eta)$ takes the form of $(A_{\mathrm{c}}, B_{\mathrm{c}}, C_{\mathrm{c}}, D_{\mathrm{c}})$ in Lemma 2.4.3. By the same argument in the proof of Theorem 2.4.3, the resulting triple (M_1, M_2, M_3) incorporates an Np-copy internal model of $A_{0\mathrm{a}}$ under

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Assumption 2.3.6. Since, in addition, Assumptions 2.3.1-2.3.3 hold and A_η is Hurwitz, the rest of the proof can be completed by following the steps given in the proof of Theorem 2.4.1. ■

Now, our goal is to obtain an agent-wise local sufficient condition that assures the property $a)$ of Definition 2.3.1 under some standard assumptions. For this purpose, define $\mu_i(t)$ as in Section 2.4.1 and let $\zeta_i(t) \triangleq [x_i^{\mathrm{T}}(t), \hat{x}_i^{\mathrm{T}}(t), \bar{z}_i^{\mathrm{T}}(t)]^{\mathrm{T}} \in \mathbb{R}^{n_i+n_{z_{2i}}}$,

$$A_{\mathrm{F}i} \triangleq \begin{bmatrix} A_i & B_iK_{1i} & B_iK_{2i} \\ L_iC_i & A_i+B_iK_{1i}-L_iC_i & B_iK_{2i} \\ G_{2i}C_i & G_{2i}D_iK_{1i} & G_{1i}+G_{2i}D_iK_{2i} \end{bmatrix},$$

$$B_{\mathrm{F}i} \triangleq \begin{bmatrix} 0 \\ -L_i \\ -G_{2i} \end{bmatrix}, \; C_{\mathrm{F}i} \triangleq \begin{bmatrix} C_i & D_iK_{1i} & D_iK_{2i} \end{bmatrix}.$$

Furthermore, consider (2.3), (2.12), (2.13), and (2.4) when $\omega(t) \equiv 0$. By inserting (2.11) into the considered equations, we have

$$\dot{\zeta}_i(t) = A_{\mathrm{F}i}\zeta_i(t) + B_{\mathrm{F}i}\mu_i(t), \quad \zeta_i(0) = \zeta_{i0}, \quad t \geq 0, \tag{2.52}$$

$$e_i(t) = C_{\mathrm{F}i}\zeta_i(t). \tag{2.53}$$

Remark 2.4.7 *Let $A_{\mathrm{L}i} \triangleq A_i - L_iC_i$. By performing the elementary row and column operations given in the proof of Theorem 2.4.3 on $A_{\mathrm{F}i}$, one can show that $\sigma(A_{\mathrm{F}i}) = \sigma(A_{\mathrm{f}i}) \cup \sigma(A_{\mathrm{L}i})$. Note that by Assumption 2.3.8, L_i can always be chosen such that $A_{\mathrm{L}i}$ is Hurwitz for all $i \in \mathcal{N}$. In conjunction with Remark 2.4.4, this shows that under Assumptions 2.3.4-2.3.6 and Assumption 2.3.8, it is always possible to find K_{1i}, K_{2i}, and L_i such that $A_{\mathrm{F}i}$ is Hurwitz for all $i \in \mathcal{N}$.*

Let $g_{\mathrm{F}i}(s) \triangleq C_{\mathrm{F}i}(sI - A_{\mathrm{F}i})^{-1}B_{\mathrm{F}i}$. For the dynamic output feedback case, we now state the following theorem.

Theorem 2.4.5 *Let Assumption 2.3.3 hold and $A_{\mathrm{F}i}$ be Hurwitz for all $i \in \mathcal{N}$. If*

$$\|g_{\mathrm{F}i}\|_\infty \rho(\mathcal{F}\mathcal{A}) < 1, \quad \forall i \in \mathcal{N}, \tag{2.54}$$

then the resulting A_η is Hurwitz.

Proof. It follows from Section 2.4.1 by comparing (2.52) and (2.53) with (2.32) and (2.33). ■

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2.5 Illustrative Numerical Examples

To illustrate some results from the previous section, we provide two numerical examples with different exosystems. In particular, the first (respectively, second) example presents the distributed dynamic state (respectively, output) feedback control law. For both examples, we consider five agents with the following system, input, output, and direct feedthrough matrices

$$
\begin{aligned}
A_i &= \begin{bmatrix} -1 & 1 \\ 0.2 & 0 \end{bmatrix}, B_i = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, C_i = \begin{bmatrix} 1 & 0 \end{bmatrix}, D_i = 0.1,\ i = 1,4,5, \\
A_i &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}, B_i = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, C_i = \begin{bmatrix} 1 & 0 & 0.4 \end{bmatrix}, D_i = 0,\ i = 2,3,
\end{aligned}
$$

and the augmented graph $\bar{\mathcal{G}}$ shown in Figure 2.1. With this setup, each agent satisfies Assumptions 2.3.4 and 2.3.8. It is also clear from Figure 2.1 that Assumption 2.3.3 holds. In the simulations, we set each nonzero a_{ij} to 1 and $k_i = 1, i = 1,2$. Moreover, initial conditions for the agents are given by $x_{10} = [1,\ 0.6]^{\mathrm{T}}$, $x_{20} = [-0.5,\ 0,\ -0.2]^{\mathrm{T}}$, $x_{30} = [-0.2,\ -0.3,\ 0]^{\mathrm{T}}$, $x_{40} = [0.6,\ 0]^{\mathrm{T}}$, $x_{50} = [0,\ 0.5]^{\mathrm{T}}$ and the controller states of all agents are initialized at zero.

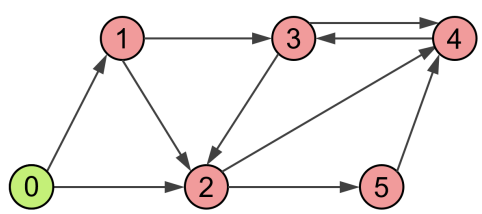

Figure 2.1: Augmented directed graph $\bar{\mathcal{G}}$.

2.5.1 Example 1

In this example, the disturbance $\delta(t)$ and the trajectory of the leader $r_0(t)$ satisfy the following dynamics

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$$
\begin{aligned}
\dot{\delta}(t) &= \begin{bmatrix} 0 & 0.01 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -0.05 \end{bmatrix} \delta(t) + \begin{bmatrix} 0 \\ 0 \\ 0.05 \end{bmatrix}, \quad \delta(0) = \begin{bmatrix} 0 \\ -0.2 \\ 0 \end{bmatrix}, \quad t \geq 0, \\
\dot{r}_0(t) &= -r_0^3(t) + u_0(t), \quad r_0(0) = 0, \quad t \geq 0,
\end{aligned}
$$

respectively, where

$$
u_0(t) = \begin{cases} 0.1t, & 0 \leq t < 100, \\ 0.1t - 2\sin(0.1t)e^{-0.01(t-100)}, & 100 \leq t < 200, \\ 14 + \sin(0.05(t-200)), & t \geq 200. \end{cases}
$$

By the solution of the disturbance dynamics with the given initial condition, $\dot{\delta}(t)$ is bounded. Since $u_0(t)$ is piecewise continuous and bounded, $r_0(t)$ is bounded by Example 4.25 in [55]; hence, $\dot{r}_0(t)$ is piecewise continuous and bounded. Clearly, $\dot{\omega}(t)$ is piecewise continuous and bounded. Furthermore, the exosystem affects the state of each agent and its tracking error through matrices

$$
\begin{aligned}
E_{\delta_1} &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \; E_{\delta_4} = \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0 & -0.1 \end{bmatrix}, \; E_{\delta_5} = \begin{bmatrix} 0 & 0 & 0 \\ -0.1 & -0.2 & 0 \end{bmatrix}, \\
E_{\delta_2} &= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0.5 \end{bmatrix}, \; E_{\delta_3} = \begin{bmatrix} 0 & -0.5 & 0 \\ 0 & 0 & -1 \\ 0 & 0.4 & 0 \end{bmatrix}, \; R_{\mathrm{r}} = 1.
\end{aligned}
$$

Suppose the piecewise continuity and boundedness of $\dot{\omega}(t)$ are the only information that we know about the exosystem. As it is suggested in the part a) of Remark 2.4.2, we then let $A_0 = 0$ and $(G_{1i}, G_{2i}) = (0,1)$ for all $i \in \mathcal{N}$. Thus, Assumptions 2.3.1, 2.3.2, 2.3.5, and 2.3.6 hold. With the following controller parameters

$$
\begin{aligned}
K_{1i} &= -\begin{bmatrix} 1.1960 & 0.9611 \end{bmatrix}, \; K_{2i} = -1.4142, \; i = 1,4,5, \\
K_{1i} &= -\begin{bmatrix} 4.2328 & 5.3904 & 1.4038 \\ 1.2604 & 1.4038 & 1.7115 \end{bmatrix}, \; K_{2i} = -\begin{bmatrix} 1.2788 \\ 1.3655 \end{bmatrix}, \; i = 2,3,
\end{aligned}
$$

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A_{fi} is Hurwitz for all $i \in \mathcal{N}$ and the condition given by (2.36) is satisfied. Thus, A_g is Hurwitz by Theorem 2.4.2. As Theorem 2.4.1 promises, ultimately bounded tracking error is observed in Figure 2.2.

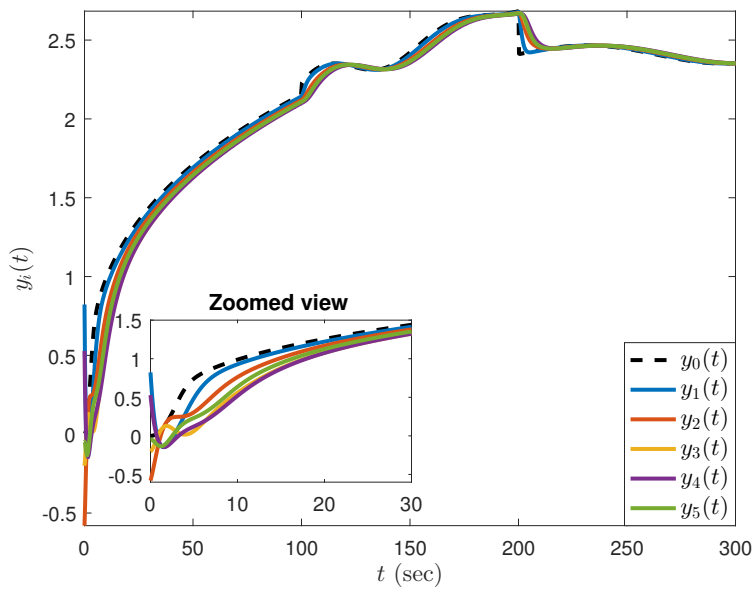

Figure 2.2: Output responses of the agents in Example 1.

2.5.2 Example 2

The disturbance and the trajectory of the leader satisfy

$$\begin{aligned}
\dot{\delta}(t) &= e^{-0.1t}, \quad \delta(0) = 1, \quad t \geq 0, \\
\dot{r}_0(t) &= \begin{bmatrix} 0 & 0.5 \\ -0.5 & 0 \end{bmatrix} r_0(t) + \begin{bmatrix} te^{-t}\sin(t) \\ 2e^{-t} \end{bmatrix}, \; r_0(0) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \; t \geq 0,
\end{aligned}$$

respectively. Moreover, $E_{\delta_1} = [1\ 0]^{\mathrm{T}}$, $E_{\delta_2} = [0\ 1\ 0]^{\mathrm{T}}$, $E_{\delta_3} = [-1.5\ 0\ 0.3]^{\mathrm{T}}$, $E_{\delta_4} = [0\ 2]^{\mathrm{T}}$, $E_{\delta_5} = [0.2\ \ -0.2]^{\mathrm{T}}$, and $R_{\mathrm{r}} = [1\ 0]$.

Suppose the unforced parts of the given dynamics are available to a control designer and the forcing terms are known to be piecewise continuous and convergent to zero. Then, let

$$A_0 = \begin{bmatrix} 0 & 0.5 & 0 \\ -0.5 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

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and

$$G_{1i} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -0.25 & 0 \end{bmatrix}, \; G_{2i} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \forall i \in \mathcal{N}.$$

Hence, Assumptions 2.3.1, 2.3.5, and 2.3.6 hold. In addition, $\lim_{t\to\infty} A_0\omega(t) - \dot{\omega}(t) = 0$. Note that Assumption 2.3.2 automatically holds since $A_0\omega(t) - \dot{\omega}(t)$ is piecewise continuous and convergent. With the following controller parameters

$$\begin{aligned}
K_{1i} &= -\begin{bmatrix} 5.1794 & 0.7932 \end{bmatrix}, \; L_i = \begin{bmatrix} 17 & 80.2 \end{bmatrix}^{\mathrm{T}}, \\
K_{2i} &= -\begin{bmatrix} 2 & 5.4458 & 10.3182 \end{bmatrix}, \; i = 1,4,5, \\
K_{1i} &= -\begin{bmatrix} 6.1916 & 5.7686 & 1.7835 \\ 3.9299 & 1.7835 & 2.4282 \end{bmatrix}, \; L_i = \begin{bmatrix} -187 & 756 & 600 \end{bmatrix}^{\mathrm{T}}, \\
K_{2i} &= -\begin{bmatrix} 0.4513 & 0.9173 & 3.3839 \\ 0.8924 & 2.2285 & 5.6377 \end{bmatrix}, \; i = 2,3,
\end{aligned}$$

$A_{\mathrm{F}i}$ is Hurwitz for all $i \in \mathcal{N}$ and the condition given by (2.54) is satisfied. Thus, A_η is Hurwitz by Theorem 2.4.5. Furthermore, it is guaranteed by Theorem 2.4.4 that $\lim_{t\to\infty} e_i(t) = 0$, $\forall i \in \mathcal{N}$ and this fact is demonstrated in Figure 2.3.

2.6 Conclusion

In this paper, we studied the cooperative output regulation problem of heterogeneous linear time-invariant multiagent systems over fixed directed communication graph topologies. Specifically, we introduced a new definition of the linear cooperative output regulation problem (see Definition 2.3.1), which allows a broad class of functions to be tracked and rejected by a network of agents, and focused on an internal model based distributed control approach. For the three different distributed control laws (i.e., dynamic state feedback, dynamic output feedback with local measurement, and dynamic output feedback), we investigated the solvability of this problem, which resulted in global and local sufficient conditions (see Theorems 2.4.1-2.4.5). In addition, the provided two numerical examples illustrated the efficacy of our

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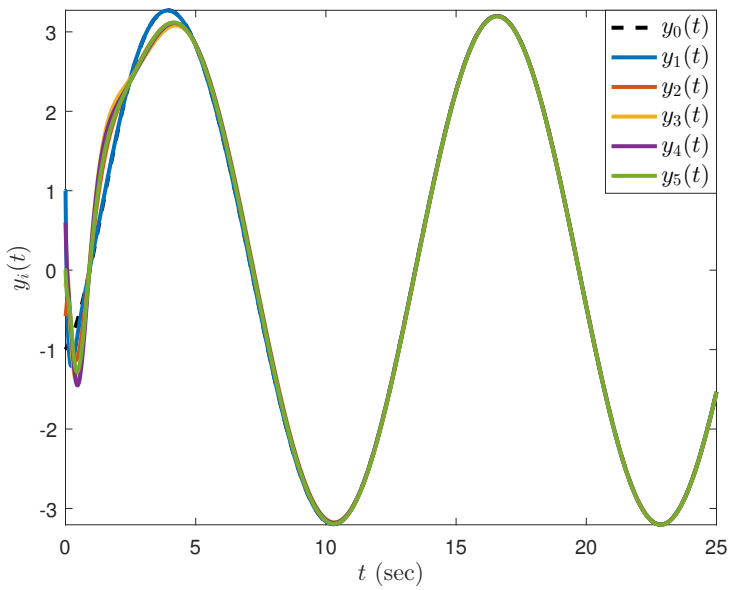

Figure 2.3: Output responses of the agents in Example 2.

contributions. Finally, we reported and addressed a considerable number of gaps in the existing related literature (see Appendices A and B and Section 2.4.1).

2.7 Acknowledgment

The authors would like to thank Dr. Chao Huang for helpful responses to our questions regarding the results in [1].

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Chapter 3: Linear Cooperative Output Regulation with Heterogeneity in Synchronization Roles[10]

This paper introduces a new definition of the linear cooperative output regulation problem in order to allow the common output synchronization (regulation) together with an additional output synchronization for a proper subset of all agents. The solvability of this problem with an internal model based distributed dynamic state feedback control law is first investigated based on a global condition. An agent-wise local sufficient condition is then presented under standard assumptions. A numerical example is finally provided to illustrate the considered problem and the proposed approach in this paper.

3.1 Introduction

3.1.1 Related Literature and Motivation

Distributed control of heterogeneous multiagent systems, which are formed by networks of agents having nonidentical dynamics and dimensions, has emerged as an attractive research direction in the last decade. In particular, the *common output synchronization (regulation)* problem of a network of heterogeneous (in dynamics and dimension) linear time-invariant systems is investigated for both the cases without and with a leader (see [1, 22, 34, 42, 43, 51, 64] and the references therein). Although the approaches in these papers differ from each other, their common denominator is that the *common output* of all agents synchronize to a *common trajectory*. Here, a *common output* of interest stands for the output variables that have the *same* physical meaning for *all* agents; hence, we shall refer to the *common output synchronization* as the *primary synchronization role* of multiagent systems. Thus, the existing literature addresses the *primary synchronization role* of multiagent systems.

From a practical standpoint, however, some output variables of a *proper subset* of all agents can share the *same* physical meaning in addition to a common output of interest. As a consequence, the following question immediately arises: *How do these specific agents achieve output synchronization not only for the common output but also for the additional output variables they have in common without deteriorating the*

[10]This chapter is previously presented in [63]. Permission is included in Appendix E.

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common output synchronization of the remaining agents? To the best of our knowledge, this question has *not* been raised or reported in the existing heterogeneous multiagent systems literature. More importantly, it yields *multiple secondary synchronization roles* of multiagent systems and related fundamental research problems to be studied.

To elucidate one possible problem, consider, for example, a network of heterogeneous dynamical systems which consists of a leader and two different groups of follower agents; see the graph $\bar{\mathcal{G}}$ in Figure 3.1. Specifically, the circle labeled with 0 denotes the leader, the circles labeled with $1, 2, 4$, and 6 denote the first group of follower agents, and the circles labeled with 3 and 5 denote the second group of follower agents. Outputs of (follower) agents in the first (two dimensional) and the second (one dimensional) group are given by $[y_a(t), y_b(t)]^{\mathrm{T}} \in \mathbb{R}^2$ and $y_a(t) \in \mathbb{R}$, respectively. If the trajectory of the leader is given by $[y_a(t), y_b(t)]^{\mathrm{T}}$, the *primary synchronization role* of the multiagent system is the synchronization of $y_a(t)$ for all agents to $y_a(t)$ of the leader. Yet, there is a *secondary synchronization role* of this multiagent system; namely, the synchronization of $y_b(t)$ for the agents in the first group to $y_b(t)$ of the leader.

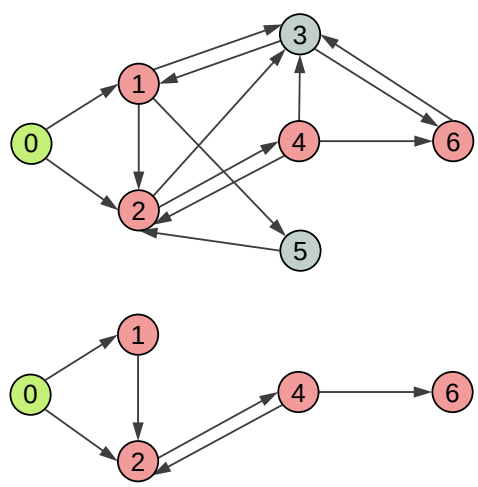

Figure 3.1: The graphs given above and below are respectively denoted by $\bar{\mathcal{G}}$ and $\bar{\mathcal{G}}_{\mathcal{S}}$. In these graphs, the circles denote the leader or the follower agents, and the arrows denote the directed edges corresponding to the physical coupling or the flow of information.

For this problem, one potential remedy is to synchronize $[y_a(t), y_b(t)]^{\mathrm{T}}$ of the agents in the first group with $[y_a(t), y_b(t)]^{\mathrm{T}}$ of the leader according to the graph $\bar{\mathcal{G}}_{\mathcal{S}}$ in Figure 3.1 and then synchronize $y_a(t)$ of the agents in the second group with $y_a(t)$ of the agents in the first group by considering only the edges from 1 to 3, 1 to 5, 2 to 3, 4 to 3, and 6 to 3 in the graph $\bar{\mathcal{G}}$. By means of this cascade approach, one can utilize the

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existing results in the literature. However, this approach *disregards* the edges from 3 to 1, 3 to 6, and 5 to 2 in the graph $\bar{\mathcal{G}}$; that is, the physical coupling or the flow of information $y_a(t)$ in the graph $\bar{\mathcal{G}}$ is partially ignored. When this simplification is not possible, the problem becomes significantly more challenging since it *cannot* be divided into two cascade synchronization problems due to the adverse effects of the ignored edges on the synchronization of $[y_a(t), y_b(t)]^{\mathrm{T}}$ for the agents in the first group with $[y_a(t), y_b(t)]^{\mathrm{T}}$ of the leader.

3.1.2 Contribution and Organization

This paper focuses on heterogeneous linear time-invariant multiagent systems with a leader when agents have heterogeneity in their synchronization roles. To this end, a new definition of the linear cooperative output regulation problem is introduced in order to allow not only the *primary output regulation* but also a *secondary output regulation* in distributed control of networks of these nonidentical agents. In particular, the solvability of this problem with an internal model based distributed dynamic state feedback control law is first investigated based on a global condition. An agent-wise local sufficient condition is then presented under standard assumptions that paves the way for independent controller design for each agent.

The organization of the remainder of this paper is as follows. Section 3.2 presents the notation and the essential mathematical preliminaries. Section 3.3 formulates the considered linear cooperative output regulation problem in this paper. The solvability of this problem by first considering a global condition and then presenting an agent-wise local sufficient condition is investigated in Section 3.4. Finally, an illustrative numerical example is provided in Section 3.5 and concluding remarks are summarized in Section 3.6.

3.2 Mathematical Preliminaries

In this paper, $\mathbb{R}$, $\mathbb{R}^n$, and $\mathbb{R}^{n\times m}$ respectively denote the sets of real numbers, $n\times 1$ real column vectors, and $n\times m$ real matrices[11]; $\mathbf{1}_n$ and I_n respectively denote the $n\times 1$ vector of all ones and the $n\times n$ identity matrix; "$\triangleq$" denotes equality by definition. In addition, we write $(\cdot)^{\mathrm{T}}$ for the transpose and $\|\cdot\|_2$ for the induced two norm of a matrix; $\sigma(\cdot)$ for the spectrum[12] and $\rho(\cdot)$ for the spectral radius of a square matrix; $(\cdot)^{-1}$ for the inverse of a nonsingular matrix; $\otimes$ for the Kronecker product; and $\mathrm{diag}(A_1,\ldots,A_n)$ for a block-diagonal matrix with matrix entries $A_1,\ldots,A_n$ on its diagonal. Finally, the space $\mathcal{L}_2$ is defined as the set of all piecewise continuous functions $u:[0,\infty)\to\mathbb{R}^m$ such that $\|u(t)\|_{\mathcal{L}_2} = \left(\int_0^\infty \|u(t)\|_2^2 \mathrm{d}t\right)^{1/2} < \infty$ [55].

[11] This paper defines all real matrices over the field of complex numbers.
[12] We follow Definition 4.4.4 in [54].

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We now concisely state the graph theoretical notation used throughout this paper, which is based on [5] and [4]. In particular, consider a fixed (i.e., time-invariant) directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{v_1, \ldots, v_N\}$ is a nonempty finite set of N nodes and $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ is a set of edges. Each node in $\mathcal{V}$ corresponds to a follower agent. There is an edge rooted at node v_j and ended at v_i (i.e., $(v_j, v_i) \in \mathcal{E}$) if and only if v_i receives information from v_j. $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N \times N}$ denotes the adjacency matrix, which describes the graph structure; that is, $a_{ij} > 0 \Leftrightarrow (v_j, v_i) \in \mathcal{E}$ and $a_{ij} = 0$ otherwise. Repeated edges and self loops are not allowed; that is, $a_{ii} = 0$, $\forall i \in \mathcal{N}$ with $\mathcal{N} = \{1, \ldots, N\}$. The set of neighbors of node v_i is denoted as $N_i = \{j \in \mathcal{V} | (v_j, v_i) \in \mathcal{E}\}$. The in-degree matrix is defined by $\mathcal{D} = \mathrm{diag}(d_1, \ldots, d_N)$ with $d_i = \sum_{j \in N_i} a_{ij}$. A directed path from node v_i to node v_j is a sequence of successive edges in the form $\{(v_i, v_p), (v_p, v_q), \ldots, (v_r, v_j)\}$. A directed graph is said to have a spanning tree if there is a root node such that it has directed paths to all other nodes in the graph. A fixed augmented directed graph is defined as $\bar{\mathcal{G}} = (\bar{\mathcal{V}}, \bar{\mathcal{E}})$, where $\bar{\mathcal{V}} = \{v_0, v_1, \ldots, v_N\}$ is the set of $N+1$ nodes, including leader node v_0 and all nodes in $\mathcal{V}$, and $\bar{\mathcal{E}} = \mathcal{E} \cup \mathcal{E}'$ is the set of edges with $\mathcal{E}'$ consisting of some edges in the form of (v_0, v_i), $i \in \mathcal{N}$.

In addition, we consider a proper subset of nodes $\mathcal{S} \subset \mathcal{V}$ such that $\mathcal{S}$ includes all follower agents that have the secondary synchronization role. Without loss of generality, the following index set is considered for $\mathcal{S}$: $\mathcal{N}_{\mathcal{S}} = \{1, \ldots, N'\}$. $\mathcal{S}$ induces a subgraph with respect to $\mathcal{G}$ and this induced subgraph is given by $\mathcal{G}_{\mathcal{S}} = (\mathcal{S}, \mathcal{E}_{\mathcal{S}})$, where $\mathcal{E}_{\mathcal{S}} = \{(v_j, v_i) \in \mathcal{E} \mid v_j, v_i \in \mathcal{S}\}$. Following the foregoing paragraph, we define adjacency and in-degree matrices for $\mathcal{G}_{\mathcal{S}}$: $\mathcal{A}_{\mathcal{S}} = [a_{sij}] \in \mathbb{R}^{N' \times N'}$ denotes the corresponding adjacency matrix, where $a_{sij} > 0 \Leftrightarrow (v_j, v_i) \in \mathcal{E}_{\mathcal{S}}$, $a_{sij} = 0$ otherwise, and $\mathcal{D}_{\mathcal{S}} = \mathrm{diag}(d_{s1}, \ldots, d_{sN'})$ with $d_{si} = \sum_{j \in N_i \cap \mathcal{N}_{\mathcal{S}}} a_{sij}$ denotes the corresponding in-degree matrix. Finally, $\bar{\mathcal{S}} = \{v_0, v_1, \ldots, v_{N'}\}$ including leader node v_0 and all nodes in $\mathcal{V}$ that have the secondary synchronization role induces a subgraph with respect to $\bar{\mathcal{G}}$ and this induced subgraph is given by $\bar{\mathcal{G}}_{\mathcal{S}} = (\bar{\mathcal{S}}, \bar{\mathcal{E}}_{\mathcal{S}})$, where $\bar{\mathcal{E}}_{\mathcal{S}} = \{(v_j, v_i) \in \bar{\mathcal{E}} \mid v_j, v_i \in \bar{\mathcal{S}}\}$.

Finally, the concept of the internal model, which is given in Definition 1.22 and Remark 1.24 of [25], is tailored for the purpose of this paper.

Definition 3.2.1 *Given any square matrix A_0, a pair of matrices (G_1, G_2) is said to incorporate a p-copy internal model of the matrix A_0 if G_1 and G_2 are given by*

$$G_1 = \mathrm{diag}(\beta_1, \ldots, \beta_p), \quad G_2 = \mathrm{diag}(\sigma_1, \ldots, \sigma_p), \tag{3.1}$$

where for $l = 1, \ldots, p$, $\beta_l \in \mathbb{R}^{h_l \times h_l}$ and $\sigma_l \in R^{h_l}$ satisfy the following conditions:

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i) The pair (β_l, σ_l) is controllable.

ii) The minimal polynomial of A_0 is equal to the characteristic polynomial of β_l.

3.3 Problem Formulation

3.3.1 Heterogeneous Multiagent Systems Setup

We focus on a system of N (follower) agents with heterogeneous linear time-invariant dynamics subject to disturbances over a fixed directed graph topology $\mathcal{G}$, where the dynamics of agent $i \in \mathcal{N}$ is given by

$$\dot{x}_i(t) = A_i x_i(t) + B_i u_i(t) + \delta_i(t), \; x_i(0) = x_{i0}, \; t \geq 0, \tag{3.2}$$

$$y_i(t) = C_i x_i(t) + D_i u_i(t), \tag{3.3}$$

with state $x_i(t) \in \mathbb{R}^{n_i}$, input $u_i(t) \in \mathbb{R}^{m_i}$, disturbance $\delta_i(t) \in \mathbb{R}^{n_i}$, and primary output $y_i(t) \in \mathbb{R}^p$. In addition to the primary output given by (3.3), every agent $i \in \mathcal{N}_\mathcal{S}$ has the following output equation

$$y_{\mathrm{s}i}(t) = C_{\mathrm{s}i} x_i(t) + D_{\mathrm{s}i} u_i(t), \tag{3.4}$$

where $y_{\mathrm{s}i}(t) \in \mathbb{R}^{p_\mathrm{s}}$ denotes the secondary output.

Consider now the exosystem given by

$$\dot{\omega}(t) = A_0 \omega(t), \quad \omega(0) = \omega_0, \quad t \geq 0, \tag{3.5}$$

$$y_0(t) = R\omega(t), \tag{3.6}$$

$$y_{\mathrm{s}0}(t) = R_\mathrm{s} \omega(t), \tag{3.7}$$

$$\delta_i(t) = E_i \omega(t), \tag{3.8}$$

that generates trajectories of the leader to be tracked and disturbances to be rejected by agents, where $\omega(t) \in R^q$ is the exosystem state and $y_0(t) \in \mathbb{R}^p$ (respectively, $y_{\mathrm{s}0}(t) \in \mathbb{R}^{p_\mathrm{s}}$) is the trajectory of the leader for the primary (respectively, secondary) synchronization role. It should be noted that this secondary synchronization role can be the prioritized (i.e., selected) one among multiple secondary roles.

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Next, consider the primary tracking error and the secondary tracking error respectively given by

$$e_i(t) \triangleq y_i(t) - y_0(t), \quad \forall i \in \mathcal{N}, \tag{3.9}$$

$$e_{si}(t) \triangleq y_{si}(t) - y_{s0}(t), \quad \forall i \in \mathcal{N}_{\mathcal{S}}. \tag{3.10}$$

As a consequence, we can write the dynamics of each agent and their corresponding tracking errors as

$$\dot{x}_i(t) = A_i x_i(t) + B_i u_i(t) + E_i \omega(t), \quad x_i(0) = x_{i0}, \quad t \geq 0, \tag{3.11}$$

$$e_i(t) = C_i x_i(t) + D_i u_i(t) - R\omega(t), \tag{3.12}$$

$$e_{si}(t) = C_{si} x_i(t) + D_{si} u_i(t) - R_s \omega(t). \tag{3.13}$$

If node v_i observes the leader node v_0, then there exists an edge (v_0, v_i) with weighting gain $k_i > 0$, otherwise $k_i = 0$. Thus, every agent that observes the leader has access to the primary tracking error $e_i(t)$. If, in addition, it belongs to $\mathcal{N}_{\mathcal{S}}$, then it has access to the secondary tracking error $e_{si}(t)$. Moreover, this paper assumes that each agent $i \in \mathcal{N}$ has access to its own state $x_i(t)$ and the primary relative output error; that is, $y_i(t) - y_j(t)$ for all $j \in N_i$. Each agent $i \in \mathcal{N}_{\mathcal{S}}$ also has access to the secondary relative output error; that is, $y_{si}(t) - y_{sj}(t)$ for all $j \in N_i \cap \mathcal{N}_{\mathcal{S}}$. The primary local virtual tracking error for each agent $i \in \mathcal{N}$ is defined as

$$e_{vi}(t) \triangleq \frac{1}{d_i + k_i}\Big(\sum_{j \in N_i} a_{ij}\big(y_i(t) - y_j(t)\big) + k_i\big(y_i(t) - y_0(t)\big)\Big). \tag{3.14}$$

In addition, the secondary local virtual tracking error for each agent $i \in \mathcal{N}_{\mathcal{S}}$ can be defined as

$$e_{svi}(t) \triangleq \frac{1}{d_{si} + k_i}\Big(\sum_{j \in N_i \cap \mathcal{N}_{\mathcal{S}}} a_{sij}\big(y_{si}(t) - y_{sj}(t)\big) + k_i\big(y_{si}(t) - y_{s0}(t)\big)\Big). \tag{3.15}$$

Finally, we define the distributed dynamic state feedback control law based on the available information to each agent as

$$u_i(t) = \begin{cases} K_{1i} x_i(t) + K_{2i} z_i(t) + K_{3i} z_{si}(t), \ \forall i \in \mathcal{N}_{\mathcal{S}}, & \text{(3.16a)} \\ K_{1i} x_i(t) + K_{2i} z_i(t), \ \forall i \in \mathcal{N} \setminus \mathcal{N}_{\mathcal{S}}, & \text{(3.16b)} \end{cases}$$

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$$\dot{z}_i(t) = G_{1i}z_i(t) + G_{2i}e_{\mathrm{v}i}(t), \quad z_i(0) = z_{i0}, \quad t \geq 0, \quad \forall i \in \mathcal{N}, \tag{3.17}$$

$$\dot{z}_{si}(t) = \tilde{G}_{1i}z_{si}(t) + \tilde{G}_{2i}e_{\mathrm{sv}i}(t), \quad z_{si}(0) = z_{si0}, \quad t \geq 0, \quad \forall i \in \mathcal{N}_{\mathcal{S}}, \tag{3.18}$$

where $z_i(t) \in \mathbb{R}^{n_{z_i}}$ and $z_{si}(t) \in \mathbb{R}^{n_{z_{si}}}$ are the controller states and the septuple $(K_{1i}, K_{2i}, K_{3i}, G_{1i}, G_{2i}, \tilde{G}_{1i}, \tilde{G}_{2i})$ is specified in Section 3.4.

3.3.2 Considered Cooperative Output Regulation Problem

Generalizing the definition of the linear cooperative output regulation problem in [34], the problem considered in this paper is now defined as follows.

Definition 3.3.1 *Given the system in* (3.5), (3.11)-(3.13)*, and the fixed augmented directed graphs* $\bar{\mathcal{G}}$ *and* $\bar{\mathcal{G}}_{\mathcal{S}}$*, find a distributed control law of the form* (3.16a)-(3.18) *such that:*

i) The resulting closed-loop system matrix is Hurwitz.

ii) For all ω_0*;* x_{i0}*,* z_{i0}*,* $i \in \mathcal{N}$*; and* z_{si0}*,* $i \in \mathcal{N}_{\mathcal{S}}$*;* $\lim_{t\to\infty} e_i(t) = 0$*,* $\forall i \in \mathcal{N}$ *and* $\lim_{t\to\infty} e_{si}(t) = 0$*,* $\forall i \in \mathcal{N}_{\mathcal{S}}$*.*

To solve the problem defined above, this paper makes the following assumptions.

Assumption 3.3.1 $A_0 \in \mathbb{R}^{q\times q}$ *has no eigenvalues with negative real parts.*

Assumption 3.3.2 *The fixed augmented directed graph* $\bar{\mathcal{G}}$ *has a spanning tree with the root node being the leader node.*

Assumption 3.3.3 *The fixed augmented directed graph* $\bar{\mathcal{G}}_{\mathcal{S}}$ *has a spanning tree with the root node being the leader node.*

Assumption 3.3.4 *The pair* (A_i, B_i) *is stabilizable for all* $i \in \mathcal{N}$*.*

Assumption 3.3.5 *For all* $\lambda \in \sigma(A_0)$*,*

$$\operatorname{rank}\begin{bmatrix} A_i - \lambda I_{n_i} & B_i \\ C_i & D_i \\ C_{\mathrm{s}i} & D_{\mathrm{s}i} \end{bmatrix} = n_i + p + p_{\mathrm{s}}, \quad \forall i \in \mathcal{N}_{\mathcal{S}}. \tag{3.19}$$

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Assumption 3.3.6 *For all $\lambda \in \sigma(A_0)$,*

$$\operatorname{rank}\begin{bmatrix} A_i - \lambda I_{n_i} & B_i \\ C_i & D_i \end{bmatrix} = n_i + p, \quad \forall i \in \mathcal{N} \setminus \mathcal{N}_{\mathcal{S}}. \tag{3.20}$$

Assumption 3.3.7 *The pair (G_{1i}, G_{2i}) incorporates a p-copy internal model of A_0 for all $i \in \mathcal{N}$.*

Assumption 3.3.8 *The pair $(\tilde{G}_{1i}, \tilde{G}_{2i})$ incorporates a p_{s}-copy internal model of A_0 for all $i \in \mathcal{N}_{\mathcal{S}}$.*

3.4 Solvability of the Problem

In this section, we investigate the solvability of the problem given in Definition 3.3.1. Specifically, our approach is twofold. First, the property *i*) of Definition 3.3.1 is assumed and it is shown, under mild conditions, that this implies the property *ii*) of Definition 3.3.1. Second, an agent-wise local sufficient condition (i.e., distributed criterion) is provided for the property *i*) of Definition 3.3.1 (i.e., the stability of the closed-loop system matrix) under standard assumptions.

We begin with some definitions that are used in this section to express the closed-loop systems in their equivalent compact forms. Let $\Phi \triangleq \operatorname{diag}(\Phi_1, \ldots, \Phi_N)$, $\Phi = A, B, C, D, E$; $\Psi_l \triangleq \operatorname{diag}(\Psi_{l1}, \ldots, \Psi_{lN})$, $\Psi = K, G$, $l = 1, 2$; $\tilde{G}_l \triangleq \operatorname{diag}(\tilde{G}_{l1}, \ldots, \tilde{G}_{lN'})$, $l = 1, 2$; $K_3 \triangleq \operatorname{diag}(K_{31}, \ldots, K_{3N'})$; $\bar{\phi} \triangleq \operatorname{diag}(\phi_1, \ldots, \phi_{N'})$, $\phi = B, D$; and $\bar{\psi}_{\mathrm{s}} \triangleq \operatorname{diag}(\psi_{\mathrm{s}1}, \ldots, \psi_{\mathrm{s}N'})$, $\psi = C, D$. Furthermore, let $x(t) \triangleq [x_1^{\mathrm{T}}(t), \ldots, x_N^{\mathrm{T}}(t)]^{\mathrm{T}} \in \mathbb{R}^{\bar{n}}$, $z(t) \triangleq [z_1^{\mathrm{T}}(t), \ldots, z_N^{\mathrm{T}}(t)]^{\mathrm{T}} \in \mathbb{R}^{\bar{n}_z}$, $z_{\mathrm{s}}(t) \triangleq [z_{\mathrm{s}1}^{\mathrm{T}}(t), \ldots, z_{\mathrm{s}N'}^{\mathrm{T}}(t)]^{\mathrm{T}} \in \mathbb{R}^{\bar{n}_{z_{\mathrm{s}}}}$, where $\bar{n} = \sum_{i=1}^{N} n_i$, $\bar{n}_z = \sum_{i=1}^{N} n_{z_i}$, $\bar{n}_{z_{\mathrm{s}}} = \sum_{i=1}^{N'} n_{z_{\mathrm{s}i}}$; $e(t) \triangleq [e_1^{\mathrm{T}}(t), \ldots, e_N^{\mathrm{T}}(t)]^{\mathrm{T}} \in \mathbb{R}^{Np}$, $e_{\mathrm{v}}(t) \triangleq [e_{\mathrm{v}1}^{\mathrm{T}}(t), \ldots, e_{\mathrm{v}N}^{\mathrm{T}}(t)]^{\mathrm{T}} \in \mathbb{R}^{Np}$, $e_{\mathrm{s}}(t) \triangleq [e_{\mathrm{s}1}^{\mathrm{T}}(t), \ldots, e_{\mathrm{s}N'}^{\mathrm{T}}(t)]^{\mathrm{T}} \in \mathbb{R}^{N'p_{\mathrm{s}}}$, $e_{\mathrm{sv}}(t) \triangleq [e_{\mathrm{sv}1}^{\mathrm{T}}(t), \ldots, e_{\mathrm{sv}N'}^{\mathrm{T}}(t)]^{\mathrm{T}} \in \mathbb{R}^{N'p_{\mathrm{s}}}$. Finally, let $\omega_{\mathrm{a}}(t) \triangleq \mathbf{1}_N \otimes \omega(t) \in \mathbb{R}^{Nq}$, $A_{0\mathrm{a}} \triangleq I_N \otimes A_0$, $R_{\mathrm{a}} \triangleq I_N \otimes R$, and $\bar{R}_{\mathrm{sa}} \triangleq I_{N'} \otimes R_{\mathrm{s}}$.

Observing $y_i(t) - y_j(t) = e_i(t) - e_j(t)$ and $y_{\mathrm{s}i}(t) - y_{\mathrm{s}j}(t) = e_{\mathrm{s}i}(t) - e_{\mathrm{s}j}(t)$, and recalling $d_i = \sum_{j \in N_i} a_{ij}$, $i \in \mathcal{N}$ and $d_{\mathrm{s}i} = \sum_{j \in N_i \cap \mathcal{N}_{\mathcal{S}}} a_{\mathrm{s}ij}$, $i \in \mathcal{N}_{\mathcal{S}}$, the expressions given by (3.14) and (3.15) can be rewritten as

$$e_{\mathrm{v}i}(t) = e_i(t) - \frac{1}{d_i + k_i} \sum_{j \in N_i} a_{ij} e_j(t), \tag{3.21}$$

$$e_{\mathrm{sv}i}(t) = e_{\mathrm{s}i}(t) - \frac{1}{d_{\mathrm{s}i} + k_i} \sum_{j \in N_i \cap \mathcal{N}_{\mathcal{S}}} a_{\mathrm{s}ij} e_{\mathrm{s}j}(t), \tag{3.22}$$

respectively. Let $\mathcal{F} \triangleq \operatorname{diag}\left(\frac{1}{d_1+k_1}, \ldots, \frac{1}{d_N+k_N}\right)$, $\mathcal{F}_{\mathcal{S}} \triangleq \operatorname{diag}\left(\frac{1}{d_{\mathrm{s}1}+k_1}, \ldots, \frac{1}{d_{\mathrm{s}N'}+k_{N'}}\right)$, $\mathcal{W} \triangleq (I_N - \mathcal{F}\mathcal{A}) \otimes I_p$, and $\mathcal{W}_{\mathcal{S}} \triangleq (I_{N'} - \mathcal{F}_{\mathcal{S}}\mathcal{A}_{\mathcal{S}}) \otimes I_{p_{\mathrm{s}}}$. Here, it should be noted that $d_i + k_i > 0$, $\forall i \in \mathcal{N}$ and $d_{\mathrm{s}i} + k_i > 0$, $\forall i \in \mathcal{N}_{\mathcal{S}}$ by

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Assumption 3.3.2 and Assumption 3.3.3, respectively; hence, $\mathcal{F}$ and $\mathcal{F}_{\mathcal{S}}$ are well-defined. From (3.21) and (3.22), one can respectively write

$$e_{\mathrm{v}}(t) = \mathcal{W}e(t), \tag{3.23}$$

$$e_{\mathrm{sv}}(t) = \mathcal{W}_{\mathcal{S}}e_{\mathrm{s}}(t). \tag{3.24}$$

Next, inserting (3.16a) and (3.16b) into (3.11) and (3.12), (3.16a) into (3.13), and using the above definitions, one can compactly write (3.11), (3.17), (3.18), (3.12), and (3.13) as

$$\dot{x}(t) = (A+BK_1)x(t)+BK_2z(t)+\tilde{B}K_3z_{\mathrm{s}}(t)+E\omega_{\mathrm{a}}(t), \quad x(0)=x_0, \quad t\geq 0, \tag{3.25}$$

$$\dot{z}(t) = G_1z(t)+G_2e_{\mathrm{v}}(t), \quad z(0)=z_0, \quad t\geq 0, \tag{3.26}$$

$$\dot{z}_{\mathrm{s}}(t) = \tilde{G}_1z_{\mathrm{s}}(t)+\tilde{G}_2e_{\mathrm{sv}}(t), \quad z_{\mathrm{s}}(0)=z_{\mathrm{s}0}, \quad t\geq 0, \tag{3.27}$$

$$e(t) = (C+DK_1)x(t)+DK_2z(t)+\tilde{D}K_3z_{\mathrm{s}}(t)-R_{\mathrm{a}}\omega_{\mathrm{a}}(t), \tag{3.28}$$

$$e_{\mathrm{s}}(t) = (C_{\mathrm{s}}+D_{\mathrm{s}}K_1)x(t)+D_{\mathrm{s}}K_2z(t)+\bar{D}_{\mathrm{s}}K_3z_{\mathrm{s}}(t)-R_{\mathrm{sa}}\omega_{\mathrm{a}}(t), \tag{3.29}$$

where $\tilde{B}=[\bar{B}^{\mathrm{T}}\ 0]^{\mathrm{T}}$, $\tilde{D}=[\bar{D}^{\mathrm{T}}\ 0]^{\mathrm{T}}$, $C_{\mathrm{s}}=[\bar{C}_{\mathrm{s}}\ 0]$, $D_{\mathrm{s}}=[\bar{D}_{\mathrm{s}}\ 0]$, and $R_{\mathrm{sa}}=[\bar{R}_{\mathrm{sa}}\ 0]$. Now, insert (3.28) into (3.23) (respectively, (3.29) into (3.24)) and replace the obtained expression with the one in (3.26) (respectively, (3.27)). Define $x_{\mathrm{g}}(t)\triangleq[x^{\mathrm{T}}(t),z^{\mathrm{T}}(t),z_{\mathrm{s}}^{\mathrm{T}}(t)]^{\mathrm{T}}\in\mathbb{R}^{\bar{n}+\bar{n}_z+\bar{n}_{z_{\mathrm{s}}}}$ and $e_{\mathrm{g}}\triangleq[e^{\mathrm{T}}(t),e_{\mathrm{s}}^{\mathrm{T}}(t)]^{\mathrm{T}}\in\mathbb{R}^{Np+N'p_{\mathrm{s}}}$. The closed-loop system given by (3.11)-(3.18) then becomes

$$\dot{x}_{\mathrm{g}}(t) = A_{\mathrm{g}}x_{\mathrm{g}}(t)+B_{\mathrm{g}}\omega_{\mathrm{a}}(t), \quad x_{\mathrm{g}}(0)=x_{\mathrm{g}0}, \quad t\geq 0, \tag{3.30}$$

$$e_{\mathrm{g}}(t) = C_{\mathrm{g}}x_{\mathrm{g}}(t)+D_{\mathrm{g}}\omega_{\mathrm{a}}(t), \tag{3.31}$$

where

$$A_{\mathrm{g}} = \begin{bmatrix} A+BK_1 & BK_2 & \tilde{B}K_3 \\ G_2\mathcal{W}(C+DK_1) & G_1+G_2\mathcal{W}DK_2 & G_2\mathcal{W}\tilde{D}K_3 \\ \tilde{G}_2\mathcal{W}_{\mathcal{S}}(C_{\mathrm{s}}+D_{\mathrm{s}}K_1) & \tilde{G}_2\mathcal{W}_{\mathcal{S}}D_{\mathrm{s}}K_2 & \tilde{G}_1+\tilde{G}_2\mathcal{W}_{\mathcal{S}}\bar{D}_{\mathrm{s}}K_3 \end{bmatrix}, \ B_{\mathrm{g}} = \begin{bmatrix} E \\ -G_2\mathcal{W}R_{\mathrm{a}} \\ -\tilde{G}_2\mathcal{W}_{\mathcal{S}}R_{\mathrm{sa}} \end{bmatrix},$$

$$C_{\mathrm{g}} = \begin{bmatrix} C+DK_1 & DK_2 & \tilde{D}K_3 \\ C_{\mathrm{s}}+D_{\mathrm{s}}K_1 & D_{\mathrm{s}}K_2 & \bar{D}_{\mathrm{s}}K_3 \end{bmatrix}, \ D_{\mathrm{g}} \begin{bmatrix} -R_{\mathrm{a}} \\ -R_{\mathrm{sa}} \end{bmatrix}.$$

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The next lemma plays a crucial role on the solvability of the problem, which is presented in Theorem 3.4.1 by assuming the property i) of Definition 3.3.1. Due to the page limitation, the proof of the next lemma is omitted.

Lemma 3.4.1 *Let Assumptions 3.3.1-3.3.3, 3.3.7, and 3.3.8 hold. If $A_{\rm g}$ is Hurwitz, then the matrix equations*

$$X_{\rm g}A_{0\rm a} = A_{\rm g}X_{\rm g}+B_{\rm g}, \tag{3.32}$$

$$0 = C_{\rm g}X_{\rm g}+D_{\rm g}, \tag{3.33}$$

have a unique solution $X_{\rm g}$.

Theorem 3.4.1 *Let Assumptions 3.3.1-3.3.3, 3.3.7, and 3.3.8 hold. If $A_{\rm g}$ is Hurwitz, then the distributed dynamic state feedback control given by* (3.16a)-(3.18) *solves the problem in Definition 3.3.1.*

Proof. Under the given conditions, (3.32) and (3.33) have a unique solution $X_{\rm g}$ by Lemma 3.4.1. It now can be derived from Lemma 1.4 in [25] that for all ω_0; x_{i0}, z_{i0}, $i\in\mathcal{N}$; and $z_{{\rm s}i0}$, $i\in\mathcal{N}_{\mathcal{S}}$; $\lim_{t\to\infty}e_{\rm g}(t)=0$. ■

Next, we focus on deriving an agent-wise local sufficient condition that assures the property i) of Definition 3.3.1 (i.e., $A_{\rm g}$ is Hurwitz) under some standard assumptions. For now, let $\xi_i(t) \triangleq [x_i^{\rm T}(t),\ z_i^{\rm T}(t),\ z_{{\rm s}i}^{\rm T}(t)]^{\rm T}\in\mathbb{R}^{n_i+n_{z_i}+n_{z_{{\rm s}i}}}$, $\forall i\in\mathcal{N}_{\mathcal{S}}$; $\xi_i(t)\triangleq[x_i^{\rm T}(t),\ z_i^{\rm T}(t)]^{\rm T}\in\mathbb{R}^{n_i+n_{z_i}}$, $\forall i\in\mathcal{N}\setminus\mathcal{N}_{\mathcal{S}}$; $\mu_i(t)\triangleq\frac{1}{d_i+k_i}\sum_{j\in N_i}a_{ij}(e_j(t)+w_j(t))$, $\forall i\in\mathcal{N}$; and $\mu_{{\rm s}i}(t)\triangleq\frac{1}{d_{{\rm s}i}+k_i}\sum_{j\in N_i\cap\mathcal{N}_{\mathcal{S}}}a_{{\rm s}ij}(e_{{\rm s}j}(t)+w_{{\rm s}j}(t))$, $\forall i\in\mathcal{N}_{\mathcal{S}}$. Here, $w_j(t)\in\mathbb{R}^p$ and $w_{{\rm s}j}(t)\in\mathbb{R}^{p_{\rm s}}$ are disturbances such that each agent $i\in\mathcal{N}$ can have access to the disturbed primary relative output error $y_i(t)-y_j(t)-w_j(t)$ for all $j\in N_i$ instead of $y_i(t)-y_j(t)$ and each agent $i\in\mathcal{N}_{\mathcal{S}}$ can have access to the disturbed secondary relative output error $y_{{\rm s}i}(t)-y_{{\rm s}j}(t)-w_{{\rm s}j}(t)$ for all $j\in N_i\cap\mathcal{N}_{\mathcal{S}}$ instead of $y_{{\rm s}i}(t)-y_{{\rm s}j}(t)$. Taking into account the disturbances $w_j(t)$ and $w_{{\rm s}j}(t)$, the primary and the secondary local virtual tracking errors in (3.14) and (3.15) are respectively written as

$$e_{{\rm v}i}(t) = e_i(t)-\mu_i(t), \tag{3.34}$$

$$e_{{\rm sv}i}(t) = e_{{\rm s}i}(t)-\mu_{{\rm s}i}(t). \tag{3.35}$$

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We now define the matrices for agent $i \in \mathcal{N}_{\mathcal{S}}$ as

$$
\begin{aligned}
A_{\mathrm{f}i} &\triangleq \begin{bmatrix} A_i+B_iK_{1i} & B_iK_{2i} & B_iK_{3i} \\ G_{2i}(C_i+D_iK_{1i}) & G_{1i}+G_{2i}D_iK_{2i} & G_{2i}D_iK_{3i} \\ \tilde{G}_{2i}(C_{\mathrm{s}i}+D_{\mathrm{s}i}K_{1i}) & \tilde{G}_{2i}D_{\mathrm{s}i}K_{2\mathrm{i}} & \tilde{G}_{1i}+\tilde{G}_{2i}D_{\mathrm{s}i}K_{3i} \end{bmatrix}, \\
\bar{B}_{\mathrm{f}i} &\triangleq \begin{bmatrix} 0 \\ -G_{2i} \\ 0 \end{bmatrix}, \bar{B}_{\mathrm{fs}i} \triangleq \begin{bmatrix} 0 \\ 0 \\ -\tilde{G}_{2i} \end{bmatrix}, B_{\mathrm{f}i} \triangleq \begin{bmatrix} \bar{B}_{\mathrm{f}i} & \bar{B}_{\mathrm{fs}i} \end{bmatrix}, \\
\bar{C}_{\mathrm{f}i} &\triangleq \begin{bmatrix} C_i+D_iK_{1i} & D_iK_{2i} & D_iK_{3i} \end{bmatrix}, \bar{C}_{\mathrm{fs}i} \triangleq \begin{bmatrix} C_{\mathrm{s}i}+D_{\mathrm{s}i}K_{1i} & D_{\mathrm{s}i}K_{2i} & D_{\mathrm{s}i}K_{3i} \end{bmatrix},
\end{aligned}
$$

and the matrices for agent $i \in \mathcal{N} \setminus \mathcal{N}_{\mathcal{S}}$ as

$$
\begin{aligned}
A_{\mathrm{f}i} &\triangleq \begin{bmatrix} A_i+B_iK_{1i} & B_iK_{2i} \\ G_{2i}(C_i+D_iK_{1i}) & G_{1i}+G_{2i}D_iK_{2i} \end{bmatrix}, B_{\mathrm{f}i} \triangleq \begin{bmatrix} 0 \\ -G_{2i} \end{bmatrix}, \\
\bar{C}_{\mathrm{f}i} &\triangleq \begin{bmatrix} C_i+D_iK_{1i} & D_iK_{2i} \end{bmatrix}.
\end{aligned}
$$

At this point, consider (3.11), (3.17), (3.18), (3.12), (3.13), (3.34), and (3.35) when $\omega(t) \equiv 0$. By inserting (3.16a) and (3.16b) into the considered equations, one can write the dynamics of each agent and its tracking error(s) respectively as

$$
\dot{\xi}_i(t) = A_{\mathrm{f}i}\xi_i(t) + B_{\mathrm{f}i}\tilde{\mu}_i(t), \quad \xi_i(0) = \xi_{i0}, \quad t \geq 0, \tag{3.36}
$$

$$
e_i(t) = \bar{C}_{\mathrm{f}i}\xi_i(t), \tag{3.37}
$$

$$
e_{\mathrm{s}i}(t) = \bar{C}_{\mathrm{fs}i}\xi_i(t), \tag{3.38}
$$

where $\tilde{\mu}_i(t) = [\mu_i^{\mathrm{T}}(t), \mu_{\mathrm{s}i}^{\mathrm{T}}(t)]^{\mathrm{T}}, \forall i \in \mathcal{N}_{\mathcal{S}}$, $\tilde{\mu}_i(t) = \mu_i(t), \forall i \in \mathcal{N} \setminus \mathcal{N}_{\mathcal{S}}$.

Let, in addition, $\mathcal{F}_1 \triangleq \mathrm{diag}\left(\frac{1}{d_1+k_1}, \ldots, \frac{1}{d_{N'}+k_{N'}}\right)$ and $\mathcal{F}_2 \triangleq \mathrm{diag}\left(\frac{1}{d_{N'+1}+k_{N'+1}}, \ldots, \frac{1}{d_N+k_N}\right)$. Partition $\mathcal{A}$ as $\mathcal{A} = [\mathcal{A}_1^{\mathrm{T}} \ \mathcal{A}_2^{\mathrm{T}}]^{\mathrm{T}}$, where $\mathcal{A}_1 \in \mathbb{R}^{N' \times N}$ and $\mathcal{A}_2 \in \mathbb{R}^{(N-N') \times N}$. Define the following matrices: $A_{\mathrm{f}} \triangleq \mathrm{diag}(A_{\mathrm{f}1}, \ldots, A_{\mathrm{f}N})$, $\bar{B}_{\mathrm{f}} \triangleq \mathrm{diag}(\bar{B}_{\mathrm{f}1}, \ldots, \bar{B}_{\mathrm{f}N'})$, $\bar{B}_{\mathrm{fs}} \triangleq \mathrm{diag}(\bar{B}_{\mathrm{fs}1}, \ldots, \bar{B}_{\mathrm{fs}N'})$, $\tilde{B}_{\mathrm{f}} \triangleq \mathrm{diag}(B_{\mathrm{f}N'+1}, \ldots, B_{\mathrm{f}N})$, $\bar{C}_{\mathrm{f}} \triangleq \mathrm{diag}(\bar{C}_{\mathrm{f}1}, \ldots, \bar{C}_{\mathrm{f}N})$, $\bar{C}_{\mathrm{fs}} \triangleq \mathrm{diag}(\bar{C}_{\mathrm{fs}1}, \ldots, \bar{C}_{\mathrm{fs}N'})$, $\tilde{C}_{\mathrm{fs}} \triangleq [\bar{C}_{\mathrm{fs}} \ 0]$, $B_{\mathrm{f}} \triangleq \begin{bmatrix} \bar{B}_{\mathrm{f}}(\mathcal{F}_1\mathcal{A}_1 \otimes I_p) & \bar{B}_{\mathrm{fs}}(\mathcal{F}_{\mathcal{S}}\mathcal{A}_{\mathcal{S}} \otimes I_{p_{\mathrm{s}}}) \\ \tilde{B}_{\mathrm{f}}(\mathcal{F}_2\mathcal{A}_2 \otimes I_p) & 0 \end{bmatrix}$, $C_{\mathrm{f}} \triangleq$

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$\begin{bmatrix} \bar{C}_{\mathrm{f}} \\ \tilde{C}_{\mathrm{fs}} \end{bmatrix}$. Let $\xi(t) \triangleq [\xi_1^{\mathrm{T}}(t), \ldots, \xi_N^{\mathrm{T}}(t)]^{\mathrm{T}} \in \mathbb{R}^{\bar{n}+\bar{n}_z+\bar{n}_{z\mathrm{s}}}$ and $w \triangleq [w_1^{\mathrm{T}}(t), \ldots, w_N^{\mathrm{T}}(t), w_{\mathrm{s}1}^{\mathrm{T}}(t), \ldots, w_{\mathrm{s}N'}^{\mathrm{T}}(T)]^{\mathrm{T}} \in \mathbb{R}^{Np+N'p_{\mathrm{s}}}$. Then, (3.36)-(3.38) can be put into the compact form given by

$$\dot{\xi}(t) = \tilde{A}_{\mathrm{f}}\xi(t) + B_{\mathrm{f}}w(t), \quad \xi(0) = \xi_0, \quad t \geq 0, \tag{3.39}$$

$$e_{\mathrm{g}}(t) = C_{\mathrm{f}}\xi(t), \tag{3.40}$$

where $\tilde{A}_{\mathrm{f}} = A_{\mathrm{f}} + B_{\mathrm{f}}C_{\mathrm{f}}$. By construction, it is clear that A_{g} is similar to $\tilde{A}_{\mathrm{f}}$; hence, they have the same eigenvalues.

By applying a version of the small gain theorem from Theorem 6.2.2.12 in [57], one can derive the agent-wise local sufficient condition given by (3.43) for $\mathcal{L}_2$ stability of the dynamics in (3.39) and (3.40). To conclude from its input-output stability that A_{g} is Hurwitz, the stabilizability and the detectability of the system of interest must be ensured. It is easy to see that if A_{f} is Hurwitz, then the pair $(\tilde{A}_{\mathrm{f}}, B_{\mathrm{f}})$ is stabilizable and the pair $(\tilde{A}_{\mathrm{f}}, C_{\mathrm{f}})$ is detectable. Therefore, the stabilizability and the detectability of the dynamics given by (3.39) and (3.40) are guaranteed if $A_{\mathrm{f}i}$ is Hurwitz for all $i \in \mathcal{N}$.

Remark 3.4.1 *For agent $i \in \mathcal{N}_{\mathcal{S}}$, let $\bar{G}_{li} \triangleq \mathrm{diag}(G_{li}, \tilde{G}_{li})$, $l = 1,2$; $\bar{K}_i \triangleq [K_{1i}\ K_{2i}\ K_{3i}]$; and $\bar{\psi}_i = [\psi_i^{\mathrm{T}}\ \psi_{\mathrm{s}i}^{\mathrm{T}}]^{\mathrm{T}}$, $\psi = C, D$. Then, $A_{\mathrm{f}i} = \bar{A}_i + \bar{B}_i\bar{K}_i$, where*

$$\bar{A}_i = \begin{bmatrix} A_i & 0 \\ \bar{G}_{2i}\bar{C}_i & \bar{G}_{1i} \end{bmatrix}, \ \bar{B}_i = \begin{bmatrix} B_i \\ \bar{G}_{2i}\bar{D}_i \end{bmatrix}.$$

Note that the pair $(\bar{G}_{1i}, \bar{G}_{2i})$ incorporates a $(p+p_{\mathrm{s}})$-copy internal model of A_0 under Assumptions 3.3.7 and 3.3.8. By Lemma 1.26 in [25], Assumptions 3.3.4, 3.3.5, 3.3.7, and 3.3.8 ensure the stabilizability of the pair $(\bar{A}_i, \bar{B}_i)$ for all $i \in \mathcal{N}_{\mathcal{S}}$. Thus, $\bar{K}_i$ can always be chosen such that $A_{\mathrm{f}i}$ is Hurwitz for all $i \in \mathcal{N}_{\mathcal{S}}$. Similarly, for agent $i \in \mathcal{N} \setminus \mathcal{N}_{\mathcal{S}}$, let $\bar{K}_i \triangleq [K_{1i}\ K_{2i}]$. Then, $A_{\mathrm{f}i} = \bar{A}_i + \bar{B}_i\bar{K}_i$, where

$$\bar{A}_i = \begin{bmatrix} A_i & 0 \\ G_{2i}C_i & G_{1i} \end{bmatrix}, \ \bar{B}_i = \begin{bmatrix} B_i \\ G_{2i}D_i \end{bmatrix}.$$

Assumptions 3.3.4, 3.3.6, and 3.3.7 guarantee the stabilizability of the pair $(\bar{A}_i, \bar{B}_i)$ for all $i \in \mathcal{N} \setminus \mathcal{N}_{\mathcal{S}}$ by Lemma 1.26 in [25]. Hence, it is always possible to find $\bar{K}_i$ such that $A_{\mathrm{f}i}$ is Hurwitz for all $i \in \mathcal{N} \setminus \mathcal{N}_{\mathcal{S}}$.

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Let $g_i(s) = \bar{C}_{\mathrm{f}i}(sI - A_{\mathrm{f}i})^{-1}B_{\mathrm{f}i}$, $\forall i \in \mathcal{N}$ and $g_{\mathrm{s}i}(s) = \bar{C}_{\mathrm{fs}i}(sI - A_{\mathrm{f}i})^{-1}B_{\mathrm{f}i}$, $\forall i \in \mathcal{N}_{\mathcal{S}}$. If we make $A_{\mathrm{f}i}$ Hurwitz for all $i \in \mathcal{N}$, we can then conclude from Corollary 5.2 in [55] that for all $i \in \mathcal{N}$, the system given by (3.36) and (3.37) is $\mathcal{L}_2$ stable with finite gain; and for all $i \in \mathcal{N}_{\mathcal{S}}$, so is the system given by (3.36) and (3.38). It follows from Theorem 5.4 in [55] that the corresponding $\mathcal{L}_2$ gains of the systems are

$$\gamma_i = \sup_{\omega \in \mathbb{R}} \|g_i(j\omega)\|_2 < \infty, \quad \forall i \in \mathcal{N}, \tag{3.41}$$

$$\gamma_{\mathrm{s}i} = \sup_{\omega \in \mathbb{R}} \|g_{\mathrm{s}i}(j\omega)\|_2 < \infty, \quad \forall i \in \mathcal{N}_{\mathcal{S}}. \tag{3.42}$$

Let $\Gamma_1 \triangleq \mathrm{diag}(\gamma_1, \ldots, \gamma_{N'})$, $\Gamma_2 \triangleq \mathrm{diag}(\gamma_{N'+1}, \ldots, \gamma_N)$, $\Gamma_3 \triangleq \mathrm{diag}(\gamma_{\mathrm{s}1}, \ldots, \gamma_{\mathrm{s}N'})$, $\Gamma \triangleq \mathrm{diag}(\Gamma_1, \Gamma_2, \Gamma_3)$.

The next theorem presents an agent-wise local sufficient condition for the problem introduced in Definition 3.3.1. Due to the page limitation, the proof of Theorem 3.4.2 is omitted.

Theorem 3.4.2 *Let Assumptions 2 and 3 hold, and $A_{\mathrm{f}i}$ be Hurwitz for all $i \in \mathcal{N}$. If*

$$\rho(\Gamma)\rho(\mathcal{Q}) < 1, \tag{3.43}$$

then A_{g} is Hurwitz, where

$$\mathcal{Q} = \begin{bmatrix} \mathcal{F}_1\mathcal{A}_1 & \mathcal{F}_{\mathcal{S}}\mathcal{A}_{\mathcal{S}} \\ \mathcal{F}_2\mathcal{A}_2 & 0 \\ \mathcal{F}_1\mathcal{A}_1 & \mathcal{F}_{\mathcal{S}}\mathcal{A}_{\mathcal{S}} \end{bmatrix}.$$

Remark 3.4.2 *The inequality given by* (3.43) *is satisfied if and only if $\gamma_i\rho(\mathcal{Q}) < 1$, $\forall i \in \mathcal{N}$ and $\gamma_{\mathrm{s}i}\rho(\mathcal{Q}) < 1$, $\forall i \in \mathcal{N}_{\mathcal{S}}$. Hence, it paves the way for independent controller design for each agent. In addition, if there were no secondary synchronization roles, the third row and the second column of $\mathcal{Q}$ would not be required and the condition would become $\gamma_i\rho(\mathcal{F}\mathcal{A}) < 1$, $\forall i \in \mathcal{N}$. This special case is consistent with the result in [1] and [51].*

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3.5 Illustrative Numerical Example

To illustrate the efficacy of our contributions documented in the previous section, consider six agents with

$$
\begin{aligned}
A_i &= \begin{bmatrix} 0 & 1 & 0 & -0.5 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0.2 & 0 & 0 & 0 \end{bmatrix}, B_i = \begin{bmatrix} 0 & 1 \\ 2 & 0 \\ 0 & -1 \\ 3 & 1 \end{bmatrix}, \\
C_i &= \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}, D_i = \begin{bmatrix} 0 & 1 \end{bmatrix}, \\
C_{si} &= \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}, D_{si} = 0, \quad i = 1,2,4,6, \\
A_i &= \begin{bmatrix} 0.1 & 1 \\ 0.5 & 0 \end{bmatrix}, B_i = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \\
C_i &= \begin{bmatrix} 1 & 0 \end{bmatrix}, D_i = -0.5, \quad i = 3,5.
\end{aligned}
$$

In addition, consider an exosystem with

$$
\begin{aligned}
A_0 &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5 \\ 0 & 0 & -0.5 & 0 \end{bmatrix}, \\
R &= \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}, R_s = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}, \\
E_1 &= \begin{bmatrix} 0.5 & 0 & 0 & 1 \\ 0 & -0.8 & 0.1 & 0 \\ 0 & 0 & 0.4 & -1 \\ -0.2 & 0 & 0 & 0 \end{bmatrix}, E_2 = \begin{bmatrix} 0 & 1 & -0.5 & 0 \\ 0 & 0 & 0 & 2 \\ -1 & -0.3 & 0 & 0 \\ 2 & 1 & 0 & -1 \end{bmatrix},
\end{aligned}
$$

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$$E_4 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -2 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \; E_6 = \begin{bmatrix} 0 & 0 & 0 & 2 \\ 3 & 2 & 1 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

$$E_3 = \begin{bmatrix} 0 & -2 & 0 & 1 \\ 0.4 & 0 & 0.2 & 0 \end{bmatrix}, \; E_5 = \begin{bmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{bmatrix},$$

and the augmented directed graphs $\bar{\mathcal{G}}$ and $\bar{\mathcal{G}}_{\mathcal{S}}$ shown in Figure 3.1. In the simulations, we set each nonzero a_{ij} and a_{sij} to 1 and $k_i = 100, i = 1,2$. Moreover, initial conditions for the exosystem and the agents are given by $\omega_0 = [0,\ 0.2,\ 1,\ -1]^{\mathrm{T}}$, $x_{10} = [1,\ 0.6,\ 0,\ 0]^{\mathrm{T}}$, $x_{20} = [-0.5,\ 0,\ -0.2,\ 0]^{\mathrm{T}}$, $x_{30} = [-0.1,\ 0]^{\mathrm{T}}$, $x_{40} = [0,\ 0,\ 0.2,\ 0.1]^{\mathrm{T}}$, $x_{50} = [0,\ 0.1]^{\mathrm{T}}$, $x_{60} = [-0.5,\ 0,\ 0,\ -0.1]^{\mathrm{T}}$, and the controller states of all agents are initialized at zero. It should be noted that $\mathcal{N}_{\mathcal{S}} = \{1,2,4,6\}$[13].

With this setup, Assumptions 3.3.1-3.3.6 hold. In addition, with the following matrices

$$G_{1i} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -0.25 & 0 \end{bmatrix}, \; G_{2i} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad \forall i \in \mathcal{N},$$

and $\tilde{G}_{1i} = G_{1i}$, $\tilde{G}_{2i} = G_{2i}$, $\forall i \in \mathcal{N}_{\mathcal{S}}$, Assumptions 3.3.7 and 3.3.8 are also satisfied. Finally, with the following controller parameters

$$K_{1i} = -\begin{bmatrix} 0.2671 & 21.0962 & -4.7667 & -9.9519 \\ 3.7818 & -7.8813 & 2.3770 & 4.8829 \end{bmatrix},$$

$$K_{2i} = -\begin{bmatrix} 0.3132 & 1.4882 & 3.2623 & 3.8043 \\ 0.9497 & 4.5627 & 9.5427 & 12.6600 \end{bmatrix},$$

$$K_{3i} = -\begin{bmatrix} -0.0300 & -0.5954 & 0.2250 & -5.5793 \\ 0.0099 & 0.1969 & -0.1051 & 1.9180 \end{bmatrix}, \forall i \in \mathcal{N}_{\mathcal{S}},$$

[13] This index set does not violate the results in the paper since any generality is not lost by re-enumerating the agents (see Section 3.2).

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$$K_{1i} = -\begin{bmatrix} 163.8941 & 199.5744 \end{bmatrix},$$
$$K_{2i} = -\begin{bmatrix} 0.0316 & 0.7889 & -0.4712 & 5.4435 \end{bmatrix}, \forall i \in \mathcal{N} \backslash \mathcal{N}_{\mathcal{S}},$$

$A_{\mathrm{f}i}$ is Hurwitz for all $i \in \mathcal{N}$ and the condition given by (3.43) is satisfied. Thus, A_{g} is Hurwitz by Theorem 3.4.2.

As is theoretically expected from Theorem 3.4.1, both the primary tracking error for all $i \in \mathcal{N}$ and the secondary tracking error for all $i \in \mathcal{N}_{\mathcal{S}}$ converge to zero. This fact is numerically illustrated in Figure 3.2.

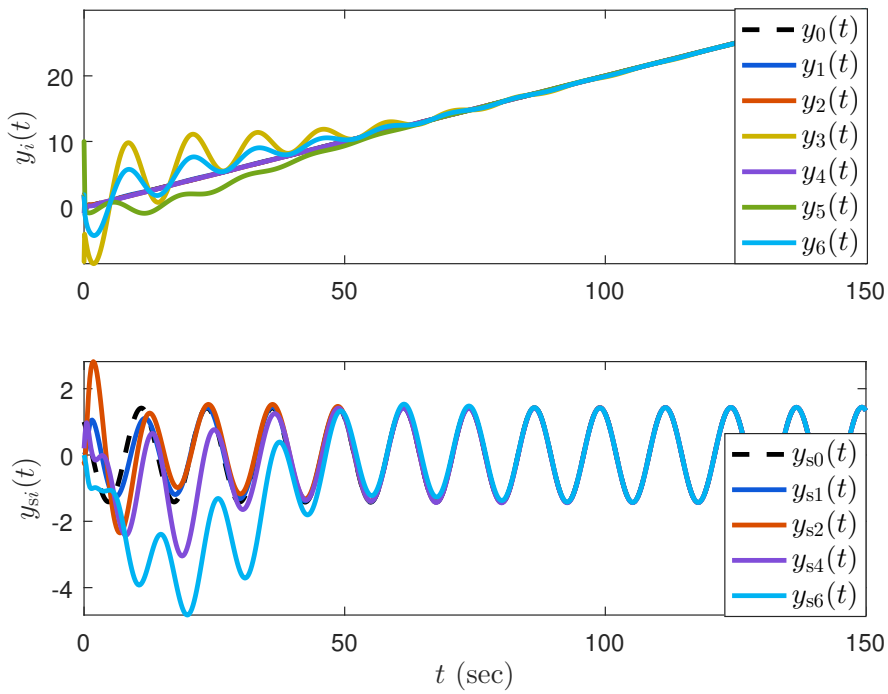

Figure 3.2: The primary output responses of the agents in $\mathcal{N}$ and the secondary output responses of the agents in $\mathcal{N}_{\mathcal{S}}$.

3.6 Conclusion

For contributing to the studies in multiagent systems, this paper introduced and addressed the *heterogeneity in synchronization roles* problem for networks of nonidentical linear time-invariant agents. Specifically, a new definition of the linear cooperative output regulation problem (Definition 3.3.1) was given to allow both the *primary output regulation* and a *secondary output regulation* in distributed control of

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multiagent systems. For an internal model based distributed state feedback control law, we first investigated the solvability of this problem based on a global condition (Theorem 3.4.1). We then provided an agent-wise local sufficient condition (Theorem 3.4.2) that paves the way for independent controller design for each agent. Future research will extend these results to multiple secondary synchronization roles of multiagent systems.

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Chapter 4: Distributed Control of Linear Multiagent Systems with Global and Local Objectives[14]

In this paper, we consider distributed control problems for high-order linear time-invariant multiagent systems with not only global but also local objectives over fixed directed communication graph topologies. The former is either leaderless synchronization or synchronization to a leader. Local objectives for a subset of agents, on the other hand, are tasks determined by agent-specific dynamical systems around the global objective. First, we construct reference models for all agents via two existing synchronization results, introduce two classes of distributed controllers, and define the considered problems. We then solve them by utilizing the converging-input converging-state property for a class of linear systems and the feedforward design methodology from the linear output regulation theory. Finally, numerical examples are presented to demonstrate the problems and the theoretical results.

4.1 Introduction

With the system-theoretic advancements in distributed control of multiagent systems over the last two decades, groups of agents are now able to utilize local information exchange for achieving a broad class of global objectives that range from synchronization (i.e., consensus) to formation (e.g., see [2–5, 46] and references therein). In particular, state synchronization in networks of identical linear systems on directed graphs has been well studied: Single-integrator and double-integrator agent dynamics are considered in [9–11] and [12, 13], respectively. For high-order linear time-invariant dynamical systems, the authors of [6, 14–18] have proposed different distributed controllers and explored conditions to guarantee leaderless synchronization. Extensions to leader-following consensus (i.e., synchronization to a leader) problems have been further investigated in [17, 19, 20].

Despite all the developments in the multiagent system literature, the following fundamental question arises: *How do some of the agents forming the multiagent system perform their own local objectives, which are defined with respect to the global objective of the multiagent system, without deteriorating the overall*

[14]This chapter has been submitted to a journal for possible publication.

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multiagent system's global objective? This question gives us another problem to address. To elucidate the problem, consider a multiagent system over a fixed directed communication graph. Specifically, assume that this system has been performing synchronization as a global objective and an operator wants to interact with a subset of the agents, for example, to drive them to an environment for data collection or battery charging purposes, or to change the frequency of the synchronization mapping for some state variables of them. When the operator injects commands to these agents for the mentioned local objectives, they are supposed to perform them and the remaining agents are expected to preserve the synchronization as if none of the agents had local objectives. After the removal of the external commands, agents that were performing their local objectives are also required to obey the synchronization.

The key points revealed by the specifications of the above scenario should be taken into account when the problem is defined and distributed controllers are proposed to tackle it. In fact, the question given in the second paragraph has been recently raised in [49] by the authors and system-theoretically addressed in [49] by providing five different distributed controllers (i.e., protocols) with comparable advantages (see Tables I and II in [49]) for single-integrator agent dynamics when the global objective is leaderless consensus. In [50], these controllers are slightly modified to achieve the leader-follower consensus as a global objective. Furthermore, several experiments are conducted on a team of ground mobile robots with these protocols. This experimental evaluation has shown that the third and fifth distributed controllers in [49] and [50] outperform the other three for both leaderless and leader-follower consensus.

This paper focuses on high-order linear time-invariant multiagent systems with both global and local objectives, where the former is either leaderless synchronization or synchronization to a leader and the latter is determined by agent-specific dynamics around the synchronization mapping of the former. Based on the existing synchronization results of [6] and [19], we construct (distributed) reference model, which achieves the global objective, for each agent. Inspired by the harmony of global and local objectives considered in [49] together with the third and fifth protocols of [49] and [50], we introduce two classes of distributed controllers and define the problems to have not only global but also local objectives for networks of linear time-invariant dynamical systems. We then propose design approaches, which utilize the feedforward design methodology from the linear output regulation theory to assign some agents local tasks, for distributed controllers to solve the problems. Finally, numerical examples are provided to illustrate our contributions.

The rest of this paper is organized as follows. In Section 4.2, we provide the notation and preliminary results. In Section 4.3, we system-theoretically state the problems. Section 4.4 contains the main

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results. Numerical examples are given in Section 4.5 and our concluding remarks are summarized in Section 4.6.

4.2 Preliminaries

4.2.1 Notation and Graph Theoretic Preliminaries

For a set S, the membership of the element s in S is denoted by $s \in S$. Let S_1 and S_2 be sets. If S_1 is a subset of S_2, we denote this by $S_1 \subset S_2$. The union and the intersection of S_1 and S_2 are denoted by $S_1 \cup S_2$ and $S_1 \cap S_2$, respectively. The complement of S_1 in S_2 is denoted by $S_2 \setminus S_1$ and the empty set is denoted by $\emptyset$. Let $\mathbb{R}$, $\mathbb{R}_{\geq 0}$, and $\mathbb{R}_{>0}$ respectively denote the sets of all real numbers, nonnegative real numbers, and positive real numbers. Let $\mathbb{C}$ be the set of all complex numbers. For $\lambda \in \mathbb{C}$, let $\mathrm{Re}(\lambda)$ denote the real part of λ. Let $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ respectively denote the sets of all $n \times 1$ real column vectors and $n \times m$ real matrices. Let $\mathbf{1}_n$ and I_n respectively denote the $n \times 1$ vector of all ones and the $n \times n$ identity matrix; and "$\triangleq$" denote equality by definition. We also write $(\cdot)^{\mathrm{T}}$ for the transpose of a matrix and $\|\cdot\|_2$ for the Euclidean norm of a vector, and $\otimes$ for the Kronecker product. Finally, $\mathrm{diag}(A_1,\ldots,A_n)$ is a block-diagonal matrix with matrix entries $A_1,\ldots,A_n$ on its diagonal.

In this paper, we consider a fixed (i.e., time-invariant) directed graph $\mathcal{G} = (\mathcal{V},\mathcal{E})$, where $\mathcal{V} = \{v_1,\ldots,v_N\}$ is a nonempty finite set of N nodes and $\mathcal{E} \subset \mathcal{V}\times\mathcal{V}$ is a set of edges. Each node in $\mathcal{V}$ corresponds to an agent. There is an edge rooted at node v_j and ended at v_i, (i.e., $(v_j,v_i) \in \mathcal{E}$), if and only if v_i receives information from v_j. In addition, $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N\times N}$ denotes the adjacency matrix of the graph $\mathcal{G}$, where $a_{ij} \in \mathbb{R}_{>0}$ if $(v_j,v_i) \in \mathcal{E}$, $a_{ij} = 0$ otherwise. Self-loops are not allowed; that is, $a_{ii} = 0$, $\forall i \in \mathcal{N}$ with $\mathcal{N} = \{1,\ldots,N\}$. The set of neighbors of node v_i is denoted as $N_i = \{j \mid (v_j,v_i) \in \mathcal{E}\}$. In-degree and Laplacian matrices of the graph $\mathcal{G}$ are defined as $\mathcal{D} = \mathrm{diag}(d_1,\ldots,d_N)$ with $d_i = \sum_{j\in N_i} a_{ij}$ and $\mathcal{L} = \mathcal{D} - \mathcal{A}$, respectively. Thus, $\mathcal{L}$ has zero row sums (i.e., $\mathcal{L}\mathbf{1}_N = 0$). A directed path from node v_i to node v_j is a sequence of successive edges in the form $(v_i,v_p),(v_p,v_q),\ldots,(v_r,v_j)$. A directed graph is said to have a spanning tree if there is a node such that it has a directed path to every other node in the graph. A fixed augmented directed graph is defined as $\bar{\mathcal{G}} = (\bar{\mathcal{V}},\bar{\mathcal{E}})$, where $\bar{\mathcal{V}} = \{v_0\} \cup \mathcal{V}$ is the set of $N+1$ nodes, including the leader node v_0 and all nodes in $\mathcal{V}$, and $\bar{\mathcal{E}} = \mathcal{E} \cup \mathcal{E}'$ is the set of edges with $\mathcal{E}'$ being a subset of $\{(v_0,v_i) \mid i \in \mathcal{N}\}$.

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4.2.2 Synchronization of Linear Systems

This subsection briefly overviews the linear quadratic regulator-based synchronization of identical linear time-invariant dynamical systems over general fixed directed communication graph topologies, where leaderless synchronization and synchronization to a leader are investigated in [6] and [19], respectively.

Consider N agents with identical linear time-invariant dynamical systems

$$\dot{x}_i(t) \;=\; Ax_i(t)+Bu_i(t), \quad x_i(0)=x_{i0}, \quad t\geq 0, \tag{4.1}$$

where $x_i(t)\in\mathbb{R}^n$ is the state and $u_i(t)\in\mathbb{R}^m$ is the input of the agent i.

Assumption 4.2.1 *The pair (A,B) is stabilizable.*

We state a well-known result of optimal control theory (e.g., see Theorem 3.4-2 of [65] together with the discussion given after the theorem) in the following lemma.

Lemma 4.2.1 *Let $Q=Q^{\mathrm{T}}\in\mathbb{R}^{n\times n}$ and $R=R^{\mathrm{T}}\in\mathbb{R}^{m\times m}$ be positive definite matrices. Suppose Assumption 4.2.1 holds. Then, the following algebraic Riccati equation*

$$A^{\mathrm{T}}P+PA+Q-PBR^{-1}B^{\mathrm{T}}P=0 \tag{4.2}$$

has a unique positive definite solution $P=P^{\mathrm{T}}\in\mathbb{R}^{n\times n}$. Therefore, $A-BK$ is Hurwitz, where $K=R^{-1}B^{\mathrm{T}}P$.

We next restate the Lemma 1 in [6] as follows:

Lemma 4.2.2 *Let $\lambda\in\mathbb{C}$ and let $K\triangleq R^{-1}B^{\mathrm{T}}P$, where P is the positive definite solution to* (4.2) *under the assumptions as in Lemma 4.2.1. If* $\mathrm{Re}(\lambda)\geq 0.5$[15]*, then $A-\lambda BK$ is Hurwitz.*

4.2.2.1 Leaderless Synchronization

It is assumed that each agent has access to the relative state information between itself and its neighbors; that is, $x_i(t)-x_j(t)$ for all $j\in N_i$. Then, consider the distributed controllers[16] given by

$$u_i(t)=cK\sum_{j\in N_i}a_{ij}\big(x_j(t)-x_i(t)\big), \tag{4.3}$$

[15] Although it is assumed in Lemma 1 of [6] that $\mathrm{Re}(\lambda)\geq 1$, its proof is still valid when $\mathrm{Re}(\lambda)\geq 0.5$.

[16] For every distributed controller considered in this paper, K is a feedback gain and c is a coupling gain as in (4.3).

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with a feedback gain $K \in \mathbb{R}^{n\times n}$ and a coupling gain $c \in \mathbb{R}_{>0}$. By inserting (4.3) into (4.1) and defining $x(t) \triangleq [x_1^{\mathrm{T}}(t), \ldots, x_N^{\mathrm{T}}(t)]^{\mathrm{T}} \in \mathbb{R}^{Nn}$, the dynamics of all agents can be compactly written as

$$\dot{x}(t) \;=\; (I_N \otimes A - c\mathcal{L} \otimes BK)x(t), \quad x(0) = x_0, \quad t \geq 0. \tag{4.4}$$

Now, we define the leaderless synchronization for the problem setup in [6].

Definition 4.2.1 *The systems in* (4.1) *with the distributed controllers of the form* (4.3) *are said to be synchronized if for all* $x_0 \in \mathbb{R}^{Nn}$*, there is a continuous mapping* $x^\star$ *of* $\mathbb{R}_{\geq 0}$ *into* $\mathbb{R}^n$ *such that* $\lim_{t\to\infty}\big(x_i(t) - x^\star(t)\big) = 0$ *for all* $i \in \mathcal{N}$.

The main result of [6] recalled in Theorem 4.2.1 shows that the feedback gain K obtained via Lemma 4.2.1 guarantees synchronization of the systems in (4.1) for any directed graph satisfying Assumption 4.2.2 provided that the coupling is strong enough, where the coupling gain c is dependent on the graph and it is determined by Lemma 4.2.2.

Assumption 4.2.2 *The fixed directed graph* $\mathcal{G}$ *with at least two nodes has a spanning tree.*

Under Assumption 4.2.2, it is known that $\mathcal{L}$ has exactly one zero eigenvalue and its other eigenvalues have positive real parts (e.g., see Lemma 3.3 in [10]). Let $\lambda_2(\mathcal{L})$ be a nonzero eigenvalue of $\mathcal{L}$ closest to the imaginary axis. Furthermore, let $w_l \in \mathbb{R}^N$ satisfy $w_l^{\mathrm{T}}\mathcal{L} = 0$ and $w_l^{\mathrm{T}}\mathbf{1}_N = 1$.

Theorem 4.2.1 *Consider the systems in* (4.1)*. Let* $K \triangleq R^{-1}B^{\mathrm{T}}P$*, where* P *is the positive definite solution to* (4.2) *under the assumptions as in Lemma 4.2.1. Let Assumption 4.2.2 hold. If* $c \geq \frac{1}{2\mathrm{Re}(\lambda_2(\mathcal{L}))}$*, then the distributed controllers in* (4.3) *guarantee that the systems in* (4.1) *are synchronized. In particular,* $x^\star(t) = (w_l^{\mathrm{T}} \otimes e^{At})x_0$.

4.2.2.2 Synchronization to a Leader

The dynamics of the leader node is given by

$$\dot{r}(t) \;=\; Ar(t), \quad r(0) = r_0, \quad t \geq 0, \tag{4.5}$$

where $r(t) \in \mathbb{R}^n$ is the state and the tracking error (i.e., $r(t) - x_i(t)$) is available to a small subset of the agents in $\mathcal{N}$. Specifically, if node v_i in $\mathcal{V}$ observes the leader node v_0, then there exists an edge (v_0, v_i) with

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weighting gain $s_i > 0$; otherwise $s_i = 0$. Each agent has also access to the relative state information. Based on the available information, every agent implements the following distributed controller[17]

$$u_i(t) = cK\Big(\sum_{j\in N_i} a_{ij}\big(x_j(t)-x_i(t)\big) + s_i\big(r(t)-x_i(t)\big)\Big). \tag{4.6}$$

By inserting (4.6) into (4.1), recalling $x(t)$ from Section 4.2.2.1, and defining $\mathcal{S} \triangleq \mathrm{diag}(s_1,\ldots,s_N)$ and $r_{\mathrm{a}}(t) \triangleq \mathbf{1}_N \otimes r(t)$, the dynamics of all agents can be written as

$$\dot{x}(t) \;=\; \big(I_N \otimes A - c(\mathcal{L}+\mathcal{S})\otimes BK\big)x(t) + \big(c(\mathcal{L}+\mathcal{S})\otimes BK\big)r_{\mathrm{a}}(t), \;\; x(0)=x_0, \;\; t\geq 0.$$

Let us define the synchronization to the leader in (4.5) for the problem in [19].

Definition 4.2.2 *The systems in* (4.1) *with the distributed controllers of the form* (4.6) *are said to synchronize to the leader in* (4.5) *if for all* $x_0 \in \mathbb{R}^{Nn}$ *and* $r_0 \in \mathbb{R}^n$, $\lim_{t\to\infty}\big(x_i(t)-r(t)\big) = 0$ *for all* $i \in \mathcal{N}$.

Assumption 4.2.3 *The fixed augmented directed graph* $\bar{\mathcal{G}}$ *has a spanning tree*[18].

Under Assumption 4.2.3, all the eigenvalues of $\mathcal{L}+\mathcal{S}$ have positive real parts (e.g., see Lemma 3.3 in [5]). Let $\lambda_1(\mathcal{L}+\mathcal{S})$ be an eigenvalue of $\mathcal{L}+\mathcal{S}$ closest to the imaginary axis. Now, Theorem 1 of [19] is stated in Theorem 4.2.2 and it is a counterpart of Theorem 4.2.1 for the synchronization to the leader in (4.5).

Theorem 4.2.2 *Consider the systems in* (4.1) *and* (4.5). *Let* $K \triangleq R^{-1}B^{\mathrm{T}}P$, *where* P *is the positive definite solution to* (4.2) *under the assumptions as in Lemma 4.2.1. Let Assumption 4.2.3 hold. If* $c \geq \frac{1}{2\mathrm{Re}(\lambda_1(\mathcal{L}+\mathcal{S}))}$, *then the distributed controllers in* (4.6) *guarantee that the systems in* (4.1) *synchronize to the leader in* (4.5).

4.2.3 Results on Boundedness and Convergence

We next concisely present useful lemmas, which are proven in Appendix C for the sake of completeness, on boundedness and convergence of piecewise continuous functions, and the converging-input converging-state property for a class of linear systems.

[17] For every distributed controller using the term $s_i(r(t)-x_i(t))$ in this paper, s_i is a weighting gain and it has the same definition as in Section 4.2.2.2.

[18] By definition of the fixed augmented directed graph given in Section 4.2.1, Assumption 4.2.3 holds if and only if the leader node has a directed path to every other node in $\bar{\mathcal{G}}$.

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Lemma 4.2.3 *Let $f:\mathbb{R}_{\geq 0}\to\mathbb{R}^n$ be a piecewise continuous*[19] *function. If $\lim_{t\to\infty} f(t)=h\in\mathbb{R}^n$, then f is bounded.*

Lemma 4.2.4 *Let $f:\mathbb{R}_{\geq 0}\to\mathbb{R}^n$ be a continuous function. Let $g:\mathbb{R}_{\geq 0}\to\{0,1\}$ be a piecewise constant*[20] *function. If $\lim_{t\to\infty} g(t)=0$, then $\lim_{t\to\infty}(gf)(t)=0$, where $(gf)(t)=g(t)f(t)$.*

Consider now the dynamical system given by

$$\dot{\zeta}(t) \quad = \quad A_{\mathrm{c}}\zeta(t)+\eta(t), \quad \zeta(0)=\zeta_0, \quad t\geq 0, \tag{4.7}$$

where $\zeta(t)\in\mathbb{R}^n$ is the state and $\eta(t)\in\mathbb{R}^n$ is the input, which is a piecewise continuous function of t.

Lemma 4.2.5 *Let A_{c} be Hurwitz. If $\lim_{t\to\infty}\eta(t)=0$, then $\lim_{t\to\infty}\zeta(t)=0$ for all $\zeta_0\in\mathbb{R}^n$.*

4.3 Problem Formulation

In this paper, we consider a system of N agents with identical dynamics given by (4.1) over a fixed directed communication graph $\mathcal{G}$. In addition, the multiagent system is subject to both global and local objectives. Specifically, the former is either leaderless synchronization or synchronization to the leader in (4.5) and the latter is associated with the tasks assigned to a subset of agents with respect to the synchronization mapping of the global objective by means of agent-specific dynamics (see Definitions 4.3.1 and 4.3.2, and Remark 4.3.2).

To this end, let $\mathcal{N}_{\mathrm{p}}$ (respectively, $\mathcal{N}_{\mathrm{p}'}$) denote the set of all agents that are (respectively, are not) assigned the local tasks, where the roles of agents are fixed (i.e., $\mathcal{N}_{\mathrm{p}}$ and $\mathcal{N}_{\mathrm{p}'}$ do not change in t). Notice that $\mathcal{N}=\mathcal{N}_{\mathrm{p}}\cup\mathcal{N}_{\mathrm{p}'}$ and $\mathcal{N}_{\mathrm{p}}\cap\mathcal{N}_{\mathrm{p}'}=\emptyset$. Now, without loss of generality, we assume the index sets as $\mathcal{N}_{\mathrm{p}}=\{1,\ldots,p\}$ and $\mathcal{N}_{\mathrm{p}'}=\{p+1,\ldots,N\}$ throughout this paper.

A subset of state variables of each agent in $\mathcal{N}_{\mathrm{p}}$ is selected through

$$y_i(t)=C_i x_i(t), \tag{4.8}$$

[19] In this paper, we follow the equivalent definitions of real vector-valued piecewise continuous functions given in page 650 of [55] and Definition 2.32 of [66].

[20] Note that every piecewise constant function is piecewise continuous, but the converse is not true.

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where $y_i(t) \in \mathbb{R}^{l_i}$, and the agent-specific dynamical system is given by

$$\dot{\delta}_i(t) = \Gamma_i \delta_i(t), \quad \delta_i(0) = \delta_{i0}, \quad t \geq 0, \tag{4.9}$$

$$\omega_i(t) = F_i \delta_i(t), \tag{4.10}$$

for each agent in $\mathcal{N}_{\mathrm{p}}$ to assign local tasks around the synchronization mapping of the global objective, where $\delta_i(t) \in \mathbb{R}^{h_i}$ is the state and $\omega_i(t) \in \mathbb{R}^{l_i}$ is the output. Let $\delta(t) \triangleq [\delta_1^{\mathrm{T}}(t), \ldots, \delta_p^{\mathrm{T}}(t)]^{\mathrm{T}} \in \mathbb{R}^{\bar{h}}$ and $\omega(t) \triangleq [\omega_1^{\mathrm{T}}(t), \ldots, \omega_p^{\mathrm{T}}(t)]^{\mathrm{T}} \in \mathbb{R}^{\bar{l}}$, where $\bar{h} = \sum_{i=1}^{p} h_i$ and $\bar{l} = \sum_{i=1}^{p} l_i$. Moreover, let $\Gamma \triangleq \mathrm{diag}(\Gamma_1, \ldots, \Gamma_p)$ and $F \triangleq \mathrm{diag}(F_1, \ldots, F_p)$. Then, the following aggregated dynamics arises from (4.9) and (4.10)

$$\dot{\delta}(t) = \Gamma \delta(t), \quad \delta(0) = \delta_0, \quad t \geq 0, \tag{4.11}$$

$$\omega(t) = F \delta(t). \tag{4.12}$$

4.3.1 Leaderless Synchronization as a Global Objective

We now introduce two classes of distributed controllers and define the problem, where the global objective is leaderless synchronization. While these controllers have the same goal (see Definition 4.3.1), they differ from each other in some ways such as required types of information and assumptions behind them, which will be highlighted in Remarks 4.3.1, 4.4.1, 4.4.3, and 4.4.4 later.

Based on Theorem 4.2.1, we consider the following reference model for each agent

$$\dot{x}_{\mathrm{r}i}(t) = A x_{\mathrm{r}i}(t) + cBK \sum_{j \in N_i} a_{ij} \big(x_{\mathrm{r}j}(t) - x_{\mathrm{r}i}(t) \big), \;\; x_{\mathrm{r}i}(0) = x_{\mathrm{r}i0}, \;\; t \geq 0. \tag{4.13}$$

By forming $x_{\mathrm{r}}(t) \triangleq [x_{\mathrm{r}1}^{\mathrm{T}}(t), \ldots, x_{\mathrm{r}N}^{\mathrm{T}}(t)]^{\mathrm{T}} \in \mathbb{R}^{Nn}$, the dynamics of all reference models can be written as

$$\dot{x}_{\mathrm{r}}(t) = (I_N \otimes A - c\mathcal{L} \otimes BK) x_{\mathrm{r}}(t), \;\; x_{\mathrm{r}}(0) = x_{\mathrm{r}0}, \;\; t \geq 0. \tag{4.14}$$

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4.3.1.1 First Distributed Controllers

The first controllers[21] are now given by (4.13) and

$$u_i(t) = cK\sum_{j\in N_i} a_{ij}\big(x_{\mathrm{r}j}(t)-x_{\mathrm{r}i}(t)\big)+l_{1i}K\big(x_{\mathrm{r}i}(t)-x_i(t)\big)+k_i\left(K\sum_{j\in N_i\setminus\mathcal{N}_{\mathrm{p}}} a_{ij}\big(x_j(t)-x_i(t)\big)+H_i\theta_i(t)\delta_i(t)\right), \quad (4.15)$$

where $H_i \in \mathbb{R}^{m\times h_i}$ is a feedforward gain and $\theta_i(t) \in \{0,1\}$ is a piecewise constant function of t used to assign the local task of the agent i; specifically, it is assigned (respectively, removed) if θ_i is set to 1 (respectively, 0). Moreover, $k_i \in \mathbb{R}_{>0}$ if $i \in \mathcal{N}_{\mathrm{p}}$, $k_i = 0$ otherwise and $l_{1i} \in \mathbb{R}_{>0}$ if $i \in \mathcal{N}_{\mathrm{p}'}$, $l_{1i} = 0$ otherwise. In (4.15), the first summation assumes that every agent exchanges its controller's state with its neighboring agents. By the definition of k_i, the terms multiplied by k_i are only effective when the agent belongs to $\mathcal{N}_{\mathrm{p}}$. Thus, each agent in $\mathcal{N}_{\mathrm{p}}$ has access to the relative state information between itself and its neighbors in $N_i \setminus \mathcal{N}_{\mathrm{p}}$ and the state of its agent-specific dynamics. Likewise, the damping term $x_{\mathrm{r}i}(t) - x_i(t)$ is only used by the agents in $\mathcal{N}_{\mathrm{p}'}$.

4.3.1.2 Second Distributed Controllers

Consider the reference models in (4.13) and remove the second summation in the output equations (4.15) of the first controllers and allow all agents in $\mathcal{N}$ to use the damping term in (4.15) as

$$u_i(t) = cK\sum_{j\in N_i} a_{ij}\big(x_{\mathrm{r}j}(t)-x_{\mathrm{r}i}(t)\big)+l_{2i}K\big(x_{\mathrm{r}i}(t)-x_i(t)\big)+k_iH_i\theta_i(t)\delta_i(t), \quad (4.16)$$

where $l_{2i} \in \mathbb{R}_{>0}$ for all $i \in \mathcal{N}$.

Remark 4.3.1 *For the agents in $\mathcal{N}_{\mathrm{p}'}$, the first and second controllers use the same types of information. On the other hand, for the agents in $\mathcal{N}_{\mathrm{p}}$, there are some differences in the required types of information: Unlike the first controllers, the second ones do not need the relative state measurement. Thus, they reduce the amount of relative information required by the first controllers. However, to implement the second controllers, every agent in $\mathcal{N}_{\mathrm{p}}$ must also be capable of measuring its own state since every controller utilizes the damping term. Hence, the second controllers make use of more self-information than the first ones. In swarm robotics, the first controllers can be preferred to the second ones since sensors measuring relative states can decrease the measurement cost.*

[21] For every introduced distributed controller in this paper, H_i is a feedforward gain, and k_i and $\theta_i(t)$ have the same definition as in the first controllers.

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4.3.1.3 Problem Definition

According to Definition 4.2.1, Theorem 4.2.1, and the harmony of the global and local objectives of the multiagent system raised in [49], the problem considered in this paper for leaderless synchronization as a global objective is defined as follows:

Definition 4.3.1 *Given the systems in* (4.1) *with Assumption 4.2.1, the fixed directed graph $\mathcal{G}$, which satisfies Assumption 4.2.2, the sets $\mathcal{N}_{\mathrm{p}}$ and $\mathcal{N}_{\mathrm{p}'}$, the output equations in* (4.8)*, and the systems in* (4.9) *and* (4.10)*, find distributed controllers of the form* (4.13) *and* (4.15) *or* (4.13) *and* (4.16) *such that for all initial conditions (i.e., $x_0 \in \mathbb{R}^{Nn}$, $x_{\mathrm{r}0} \in \mathbb{R}^{Nn}$, and $\delta_0 \in \mathbb{R}^{\bar{h}}$) of the closed-loop system, the following properties hold:*

i) $\lim_{t\to\infty}\left(x_i(t) - x_{\mathrm{r}}^{\star}(t)\right) = 0$ *for all* $i \in \mathcal{N}_{\mathrm{p}'}$*, where* $x_{\mathrm{r}}^{\star}(t) = (w_l^{\mathrm{T}} \otimes e^{At})x_{\mathrm{r}0}$*,*

ii) $\lim_{t\to\infty}\left(x_i(t) - x_{\mathrm{r}}^{\star}(t)\right) = 0$ *if* $\lim_{t\to\infty}\theta_i(t) = 0$ *for any* $i \in \mathcal{N}_{\mathrm{p}}$*,*

iii) $\lim_{t\to\infty}\Big(y_i(t) - \big(y_i^{\star}(t) + \omega_i(t)\big)\Big) = 0$ *if* $\lim_{t\to\infty}\theta_i(t) = 1$ *for any* $i \in \mathcal{N}_{\mathrm{p}}$*, where* $y_i^{\star}(t) = C_i x_{\mathrm{r}}^{\star}(t)$*.*

Remark 4.3.2 *A few notes regarding the properties given by Definition 4.3.1 are in order: The properties i) and ii) say that the agents in $\mathcal{N}_{\mathrm{p}'}$ obey the global objective of the multiagent system and any agent in $\mathcal{N}_{\mathrm{p}}$ obeys the global objective after the removal of its local objective, respectively. In addition, the property iii) specifies the local objective of any agent in $\mathcal{N}_{\mathrm{p}}$ with respect to the global objective of the multiagent system. Notice that the global objective is independent of the local objectives of the agents in $\mathcal{N}_{\mathrm{p}}$ (i.e., $x_{\mathrm{r}}^{\star}(t)$ does not depend on the agent-specific dynamics given by* (4.9) *and* (4.10)*). This feature makes the problem in Definition 4.3.1 completely different from the formation control problem studied in [67] and [68]. Specifically, the formation reference function is directly affected by the formation in [67] and [68] (e.g., see Definition 1, Theorem 2, and Remark 7 in [67]). Therefore, even if we take $C_i = I_n$ for all $i \in \mathcal{N}_{\mathrm{p}}$, the reference formation function and the formation cannot be regarded as the global objective and the local objective of the multiagent system, respectively.*

Remark 4.3.3 *The mapping $x_{\mathrm{r}}^{\star}(t)$ of the global objective is dependent on the graph topology $\mathcal{G}$, the system matrix A of each agent, and the initial values $x_{\mathrm{r}0}$ of the reference models. Therefore, $x_{\mathrm{r}0}$ can be used to modify*[22] *$x_{\mathrm{r}}^{\star}(t)$ although it cannot be specified arbitrarily. If one sets $x_{\mathrm{r}0} = x_0$, the map $x_{\mathrm{r}}^{\star}(t)$ becomes the original synchronization mapping $x^{\star}(t)$ given in Theorem 4.2.1.*

[22] Note that e^{At} is nonsingular for every $t \in \mathbb{R}_{\geq 0}$ and recall that w_l is an eigenvector of $\mathcal{L}^{\mathrm{T}}$. Thus, $w_l^{\mathrm{T}} \otimes e^{At}$ has full row rank for every $t \in \mathbb{R}_{\geq 0}$. Fix $t^* \in \mathbb{R}_{\geq 0}$. Then, for every $x_{\mathrm{r}}^* \in \mathbb{R}^n$, there exists an $x_{\mathrm{r}0} \in \mathbb{R}^{Nn}$ such that $x_{\mathrm{r}}^* = (w_l^{\mathrm{T}} \otimes e^{At^*})x_{\mathrm{r}0}$.

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4.3.2 Synchronization to a Leader as a Global Objective

Next, the term $s_i\big(r(t)-x_{\mathrm{r}i}(t)\big)$ is incorporated into the controllers of Section 4.3.1 to make them suitable for synchronization to the leader in (4.5) as a global objective.

First, based on Theorem 4.2.2, (4.13) is adjusted as follows

$$\dot{x}_{\mathrm{r}i}(t) \;=\; Ax_{\mathrm{r}i}(t)+cBK\Bigg(\sum_{j\in N_i}a_{ij}\big(x_{\mathrm{r}j}(t)-x_{\mathrm{r}i}(t)\big)+s_i\big(r(t)-x_{\mathrm{r}i}(t)\big)\Bigg),\quad x_{\mathrm{r}i}(0)=x_{\mathrm{r}i0},\quad t\geq 0. \tag{4.17}$$

4.3.2.1 First Distributed Controllers

Accordingly, instead of (4.15), we have the following output equations for the controllers

$$\begin{aligned} u_i(t) \;=\; & cK\left(\sum_{j\in N_i}a_{ij}\big(x_{\mathrm{r}j}(t)-x_{\mathrm{r}i}(t)\big)+s_i\big(r(t)-x_{\mathrm{r}i}(t)\big)\right)+l_{1i}K\big(x_{\mathrm{r}i}(t)-x_i(t)\big)\\ & +k_i\left(K\sum_{j\in N_i\setminus\mathcal{N}_{\mathrm{p}}}a_{ij}\big(x_j(t)-x_i(t)\big)+H_i\theta_i(t)\delta_i(t)\right). \end{aligned} \tag{4.18}$$

4.3.2.2 Second Distributed Controllers

In place of (4.16), we consider

$$u_i(t) \;=\; cK\left(\sum_{j\in N_i}a_{ij}\big(x_{\mathrm{r}j}(t)-x_{\mathrm{r}i}(t)\big)+s_i\big(r(t)-x_{\mathrm{r}i}(t)\big)\right)+l_{2i}K\big(x_{\mathrm{r}i}(t)-x_i(t)\big)+k_iH_i\theta_i(t)\delta_i(t). \tag{4.19}$$

4.3.2.3 Problem Definition

When the global objective of the multiagent system is synchronization to the leader in (4.5), the next definition slightly modifies Definition 4.3.1 by taking Definition 4.2.2 into account.

Definition 4.3.2 *Given the systems in* (4.1) *with Assumption 4.2.1, the system in* (4.5)*, the fixed augmented directed graph $\bar{\mathcal{G}}$, which satisfies Assumption 4.2.3, the sets $\mathcal{N}_{\mathrm{p}}$ and $\mathcal{N}_{\mathrm{p}'}$, the output equations in* (4.8)*, and the systems in* (4.9) *and* (4.10)*, find distributed controllers of the form* (4.17) *and* (4.18) *or* (4.17) *and* (4.19) *such that for all initial conditions (i.e., $x_0\in\mathbb{R}^{Nn}$, $r_0\in\mathbb{R}^n$, $x_{\mathrm{r}0}\in\mathbb{R}^{Nn}$, and $\delta_0\in\mathbb{R}^{\bar{h}}$) of the closed-loop system, the following properties hold:*

i) $\lim_{t\to\infty}\big(x_i(t)-r(t)\big)=0$ *for all* $i\in\mathcal{N}_{\mathrm{p}'}$,

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ii) $\lim_{t\to\infty}\left(x_i(t)-r(t)\right)=0$ *if* $\lim_{t\to\infty}\theta_i(t)=0$ *for any* $i\in\mathcal{N}_{\mathrm{p}}$,

iii) $\lim_{t\to\infty}\left(y_i(t)-\left(y_i^{\star}(t)+\omega_i(t)\right)\right)=0$ *if* $\lim_{t\to\infty}\theta_i(t)=1$ *for any* $i\in\mathcal{N}_{\mathrm{p}}$, *where* $y_i^{\star}(t)=C_i r(t)$.

Remark 4.3.4 *In general, the set of formation vectors for an agent in the time-varying formation tracking problem with one leader, which is investigated in Corollary 1 of [69], differs from the local objectives given in Definition 4.3.2. Essentially, the formation vector is allocated to each state variable of the agent, while the local objective can be associated with a proper subset of its state variables. It is also worth mentioning that global and local generalized disturbances introduced in [40] can be interpreted as global and local objectives in the context of leader-following consensus problems. If the distributed regulator in Theorem 3 of [40] is adjusted in such a way that the properties of Definition 4.3.2 are satisfied, then an alternative solution to the problem can be obtained.*

4.3.3 Assumptions and Lemmas from Output Regulation Theory

To satisfy the property *iii)* of Definitions 4.3.1 and 4.3.2, this paper makes the additional standard assumptions from the linear output regulation theory (e.g., see Chapter 1 of [25]) as follows:

Assumption 4.3.1 $\Gamma_i\in\mathbb{R}^{h_i\times h_i}$ *has no eigenvalues with negative real parts for all* $i\in\mathcal{N}_{\mathrm{p}}$.

Assumption 4.3.2 *For all* $i\in\mathcal{N}_{\mathrm{p}}$, *there exist* X_i *and* U_i *that satisfy the following linear matrix equations (i.e., regulator equations)*

$$X_i\Gamma_i = AX_i+BU_i, \tag{4.20}$$

$$0 = C_iX_i-F_i. \tag{4.21}$$

The next lemma immediately follows from Theorem 1.7 and Lemma 1.4 in [25].

Lemma 4.3.1 *Under Assumption 4.2.1, let the feedback gain* K_{1i} *be such that* $A+BK_{1i}$ *is Hurwitz for all* $i\in\mathcal{N}_{\mathrm{p}}$. *Let Assumptions 4.3.1 and 4.3.2 hold. If the feedforward gain* $K_{2i}=U_i-K_{1i}X_i$ *for all* $i\in\mathcal{N}_{\mathrm{p}}$, *then*

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the following linear matrix equations

$$\tilde{X}_i\Gamma_i = (A+BK_{1i})\tilde{X}_i + BK_{2i}, \tag{4.22}$$

$$0 = C_i\tilde{X}_i - F_i \tag{4.23}$$

have a unique solution $\tilde{X}_i$ *for all* $i \in \mathcal{N}_{\mathrm{p}}$.

For any agent $i \in \mathcal{N}_{\mathrm{p}}$, we consider a generic dynamical system given by

$$\dot{\tilde{\zeta}}_i(t) = (A+BK_{1i})\tilde{\zeta}_i(t) + BK_{2i}\delta_i(t) + \phi_i(t), \quad \tilde{\zeta}_i(0) = \tilde{\zeta}_{i0}, \quad t \geq 0, \tag{4.24}$$

$$\tilde{\beta}_i(t) = C_i\tilde{\zeta}_i(t) - \omega_i(t), \tag{4.25}$$

where $\tilde{\zeta}_i(t) \in \mathbb{R}^n$ is the state and $\phi_i(t) \in \mathbb{R}^n$ is the piecewise continuous input. In Section 4.4, this generic dynamical system is utilized together with the next lemma. If $\phi_i(t) = 0, \forall t \geq 0$, then Lemma 4.3.2 is a special case of Lemma 1.4 in [25], where its proof is given in Appendix C.

Lemma 4.3.2 *Consider the hypotheses of Lemma 4.3.1 and the system in* (4.24) *and* (4.25). *If* $\lim_{t\to\infty}\phi_i(t) = 0$, *then* $\lim_{t\to\infty}\tilde{\beta}_i(t) = 0$ *for all* $\tilde{\zeta}_{i0} \in \mathbb{R}^n$ *and* $\delta_{i0} \in \mathbb{R}^{h_i}$.

4.4 Main Results

This section proposes approaches to design the distributed controllers introduced in Section 4.3 for the problems stated in Definitions 4.3.1 and 4.3.2.

4.4.1 Solutions to the Problem in Definition 4.3.1

First, define the error between the state of each agent and the state of its corresponding reference model as

$$\tilde{x}_i(t) \triangleq x_i(t) - x_{ri}(t). \tag{4.26}$$

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4.4.1.1 Synthesis of First Distributed Controllers

Inserting (4.15) into (4.1) and using (4.13) together with (4.26), the error dynamics of each agent can be written as

$$\begin{aligned}\dot{\tilde{x}}_i(t) &= (A - l_{1i}BK)\tilde{x}_i(t) + k_iB\Big(K\sum_{j\in N_i\setminus\mathcal{N}_\mathrm{p}} a_{ij}\big(\tilde{x}_j(t)-\tilde{x}_i(t)+x_{\mathrm{r}j}(t)-x_{\mathrm{r}i}(t)\big)+H_i\theta_i(t)\delta_i(t)\Big),\\ &\tilde{x}_i(0)=\tilde{x}_{i0},\quad t\geq 0.\end{aligned} \tag{4.27}$$

To exploit the first controllers given by (4.13) and (4.15), the following assumption is required.

Assumption 4.4.1 *For all $i \in \mathcal{N}_\mathrm{p}$, $N_i \setminus \mathcal{N}_\mathrm{p}$ is nonempty.*

Remark 4.4.1 *Let $d_{\mathrm{p}i} \triangleq \sum_{j\in N_i\setminus\mathcal{N}_\mathrm{p}} a_{ij}$ be the another in-degree for each agent in $\mathcal{N}_\mathrm{p}$. Thus, Assumption 4.4.1 is equivalent to $d_{\mathrm{p}i} \in \mathbb{R}_{>0}$ for all $i \in \mathcal{N}_\mathrm{p}$. Here, we note that this assumption is not necessary to solve the problem stated in Definition 4.3.1; specifically, it is not required in the second controllers.*

We now state one of the main results of this paper whose overall conclusion is that the distributed controllers in (4.13) and (4.15) with an appropriate design approach solve the problem in Definition 4.3.1.

Theorem 4.4.1 *Consider the hypotheses of Theorem 4.2.1 and the controllers in* (4.13) *and* (4.15)*. If $l_{1i} \geq 0.5$ for all $i \in \mathcal{N}_{\mathrm{p}'}$, then the property i) of Definition 4.3.1 holds. If, in addition, $k_i d_{\mathrm{p}i} \geq 0.5$ for all $i \in \mathcal{N}_\mathrm{p}$, then the property ii) of Definition 4.3.1 holds. Furthermore, let Assumptions 4.3.1 and 4.3.2 hold and $H_i \triangleq k_i^{-1}U_i + d_{\mathrm{p}i}KX_i$ for all $i \in \mathcal{N}_\mathrm{p}$. Then the property iii) of Definition 4.3.1 holds.*

Proof. Let $x_{\mathrm{r}0} \in \mathbb{R}^{Nn}$ be given. By Theorem 4.2.1, $\lim_{t\to\infty}\big(x_{\mathrm{r}i}(t) - x_\mathrm{r}^\star(t)\big) = 0$ for all $i \in \mathcal{N}$. Let $x_0 \in \mathbb{R}^{Nn}$ and $\delta_0 \in \mathbb{R}^{\bar{h}}$ be given. First, fix $i \in \mathcal{N}_{\mathrm{p}'}$. Then the error dynamics in (4.27) reduces to the form given by

$$\dot{\tilde{x}}_i(t) = (A - l_{1i}BK)\tilde{x}_i(t), \quad \tilde{x}_i(0) = \tilde{x}_{i0}, \quad t \geq 0. \tag{4.28}$$

Since $l_{1i} \geq 0.5$ for all $i \in \mathcal{N}_{\mathrm{p}'}$, $A - l_{1i}BK$ is Hurwitz for all $i \in \mathcal{N}_{\mathrm{p}'}$ by Lemma 4.2.2. Hence, $\lim_{t\to\infty}\tilde{x}_i(t) = 0$. Since $\lim_{t\to\infty}\big(x_{\mathrm{r}i}(t) - x_\mathrm{r}^\star(t)\big) = 0$ and $\lim_{t\to\infty}\tilde{x}_i(t) = 0$, $\lim_{t\to\infty}\big(x_i(t) - x_\mathrm{r}^\star(t)\big) = 0$, which gives the first conclusion of the theorem.

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Second, let $i \in \mathcal{N}_{\rm p}$. Define $\rho_i(t) \triangleq \sum_{j\in N_i\setminus\mathcal{N}_{\rm p}} a_{ij}\tilde{x}_j(t)$ and $\mu_i(t) \triangleq \sum_{j\in N_i\setminus\mathcal{N}_{\rm p}} a_{ij}\big(x_{{\rm r}j}(t)-x_{{\rm r}i}(t)\big)$. Using the definition of $d_{{\rm p}i}$, (4.27) can now be written as

$$\dot{\tilde{x}}_i(t) = (A-k_id_{{\rm p}i}BK)\tilde{x}_i(t)+k_iB\Big(K\big(\rho_i(t)+\mu_i(t)\big)+H_i\theta_i(t)\delta_i(t)\Big), \quad \tilde{x}_i(0)=\tilde{x}_{i0}, \quad t\geq 0. \tag{4.29}$$

Since $k_id_{{\rm p}i}\geq 0.5$ for all $i\in\mathcal{N}_{\rm p}$, $A-k_id_{{\rm p}i}BK$ is Hurwitz for all $i\in\mathcal{N}_{\rm p}$ by Lemma 4.2.2. Note that the solution of the linear time-invariant system in (4.14) is continuous on $[0,\infty)$; that is, $x_{{\rm r}i}(t)$ is continuous on $[0,\infty)$ for all $i\in\mathcal{N}$. Thus, $\mu_i(t)$ is a continuous function of t. Similarly, $\delta_i(t)$ is a continuous function of t. By the first part of the proof, $\rho_i(t)$ is a continuous function of t and $\lim_{t\to\infty}\rho_i(t)=0$. Moreover, $\lim_{t\to\infty}\mu_i(t)=0$ since all the reference models are synchronized. For the second conclusion of the theorem, let $\lim_{t\to\infty}\theta_i(t)=0$. By Lemma 4.2.4, $\lim_{t\to\infty}H_i\theta_i(t)\delta_i(t)=0$. Clearly, $k_iB\Big(K\big(\rho_i(t)+\mu_i(t)\big)+H_i\theta_i(t)\delta_i(t)\Big)$ is a convergent piecewise continuous function of t to 0. By Lemma 4.2.5, $\lim_{t\to\infty}\tilde{x}_i(t)=0$. Hence, the property *ii)* of Definition 4.3.1 follows from the same argument in the first part of the proof. For the third conclusion of the theorem, let $\lim_{t\to\infty}\theta_i(t)=1$. Put $\tilde{e}_i(t)\triangleq C_i\tilde{x}_i(t)-\omega_i(t)$. Observe that $y_i(t)-\big(y_i^\star(t)+\omega_i(t)\big)=\tilde{e}_i(t)+C_i\big(x_{{\rm r}i}(t)-x_{\rm r}^\star(t)\big)$. Now, it suffices to show that $\lim_{t\to\infty}\tilde{e}_i(t)=0$. Let $K_{1i}\triangleq -k_id_{{\rm p}i}K$ and $K_{2i}\triangleq k_iH_i$. Then, (4.29) can be rewritten as

$$\begin{aligned}\dot{\tilde{x}}_i(t) &= (A+BK_{1i})\tilde{x}_i(t)+BK_{2i}\delta_i(t)+k_iB\Big(K\big(\rho_i(t)+\mu_i(t)\big)-H_i\big(1-\theta_i(t)\big)\delta_i(t)\Big),\\ &\tilde{x}_i(0)=\tilde{x}_{i0}, \quad t\geq 0.\end{aligned} \tag{4.30}$$

By Lemma 4.2.4, $\lim_{t\to\infty}H_i\big(1-\theta_i(t)\big)\delta_i(t)=0$. Clearly, $k_iB\Big(K\big(\rho_i(t)+\mu_i(t)\big)-H_i\big(1-\theta_i(t)\big)\delta_i(t)\Big)$ is a convergent piecewise continuous function of t to 0. Using the given definition of H_i, it can be seen that $K_{2i}=U_i-K_{1i}X_i$. All the conditions of Lemma 4.3.2 therefore hold. Hence, $\lim_{t\to\infty}\tilde{e}_i(t)=0$, as desired. ■

Remark 4.4.2 *We have not incorporated Assumption 4.4.1 into the preceding theorem explicitly. However, Assumption 4.4.1 is necessary for the following condition given in Theorem 4.4.1: $k_id_{{\rm p}i}\geq 0.5$ for all $i\in\mathcal{N}_{\rm p}$. If each nonzero a_{ij} and k_i is 1, then Assumption 4.4.1 is also sufficient.*

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4.4.1.2 Synthesis of Second Distributed Controllers

Inserting (4.16) into (4.1) and using (4.13) together with (4.26), the error dynamics of each agent can be written as

$$\dot{\tilde{x}}_i(t) \;=\; (A - l_{2i}BK)\tilde{x}_i(t) + k_i BH_i\theta_i(t)\delta_i(t), \quad \tilde{x}_i(0) = \tilde{x}_{i0}, \quad t \geq 0. \tag{4.31}$$

The distributed controllers in (4.13) and (4.16) with the following design approach address the problem in Definition 4.3.1. The proof of Theorem 4.4.2 is similar to Theorem 4.4.1; hence, we omit it.

Theorem 4.4.2 *Consider the hypotheses of Theorem 4.2.1 and the controllers in* (4.13) *and* (4.16)*. If* $l_{2i} \geq 0.5$ *for all* $i \in \mathcal{N}$*, then the properties i) and ii) of Definition 4.3.1 hold. Furthermore, let Assumptions 4.3.1 and 4.3.2 hold and* $H_i \triangleq k_i^{-1}(U_i + l_{2i}KX_i)$ *for all* $i \in \mathcal{N}_{\mathrm{p}}$*. Then the property iii) of Definition 4.3.1 holds.*

Remark 4.4.3 *In contrast to the first controllers, the feedforward gain* H_i *of the agents in* $\mathcal{N}_{\mathrm{p}}$ *for the second controllers is independent of the graph topology (i.e.,* d_{p_i}*).*

Remark 4.4.4 *Let the first and second controllers have the same feedback gain* K*. Then, for the agents in* $\mathcal{N}_{\mathrm{p}'}$*, the error dynamics due to the first and second controllers can be made identical by choosing* $l_{1i} = l_{2i}$ *for all* $i \in \mathcal{N}_{\mathrm{p}'}$*. Therefore, the responses of the agents in* $\mathcal{N}_{\mathrm{p}'}$ *with the first controllers are identical to the responses with the second controllers under the same initial conditions (see the examples in Section 4.5). Moreover, let the first and second controllers use the same solutions of the regulator equations. Under Assumption 4.4.1, for the agents in* $\mathcal{N}_{\mathrm{p}}$*, both the system matrices of the error dynamics and the feedforward gains* H_i *due to the first and second controllers can now be made identical by taking* $k_i = l_{2i}/d_{\mathrm{p}_i}$ *for all* $i \in \mathcal{N}_{\mathrm{p}}$*. With the above choices, for the agents in* $\mathcal{N}_{\mathrm{p}}$*, the term* $k_i BK(\rho_i(t) + \mu_i(t))$*, which converges to* 0*, in* (4.29) *is the only term that can yield different performances for the first and second controllers. However, under the same initial conditions, the responses of the agents in* $\mathcal{N}_{\mathrm{p}}$ *with the first controllers will become indistinguishable from the responses with the second controllers after the decay of this term (see the examples in Section 4.5).*

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4.4.2 Solutions to the Problem in Definition 4.3.2

Since the stability analyses are similar to the ones in Section 4.4.1, we omit the details and proofs. Following the steps in Section 4.4.1.1, one readily sees that (4.17) and (4.18) yield (4.27). Likewise, (4.17) and (4.19) yield (4.31). Not surprisingly, we have the solutions given in Propositions 4.4.1 and 4.4.2.

Proposition 4.4.1 *Consider the hypotheses of Theorem 4.2.2 and the controllers in* (4.17) *and* (4.18)*. If* $l_{1i} \geq 0.5$ *for all* $i \in \mathcal{N}_{\mathrm{p}'}$*, then the property i) of Definition 4.3.2 holds. If, in addition,* $k_i d_{\mathrm{p}i} \geq 0.5$ *for all* $i \in \mathcal{N}_{\mathrm{p}}$*, then the property ii) of Definition 4.3.2 holds. Furthermore, let Assumptions 4.3.1 and 4.3.2 hold and* $H_i \triangleq k_i^{-1} U_i + d_{\mathrm{p}i} K X_i$ *for all* $i \in \mathcal{N}_{\mathrm{p}}$*. Then the property iii) of Definition 4.3.2 holds.*

Proposition 4.4.2 *Consider the hypotheses of Theorem 4.2.2 and the controllers in* (4.17) *and* (4.19)*. If* $l_{2i} \geq 0.5$ *for all* $i \in \mathcal{N}$*, then the properties i) and ii) of Definition 4.3.2 hold. Furthermore, let Assumptions 4.3.1 and 4.3.2 hold and* $H_i \triangleq k_i^{-1}(U_i + l_{2i} K X_i)$ *for all* $i \in \mathcal{N}_{\mathrm{p}}$*. Then the property iii) of Definition 4.3.2 holds.*

4.5 Numerical Examples

In this section, we present numerical examples about the theoretical results provided in Section 4.4. The communication graphs $\mathcal{G}$ and $\bar{\mathcal{G}}$ described in Figure 4.1 are used for these examples. It is also assumed that $p = 3$, that is; $\mathcal{N}_{\mathrm{p}} = \{1, 2, 3\}$. With this setup, Assumptions 4.2.2, 4.2.3, and 4.4.1 are clearly satisfied. In the simulations, we take $a_{ij} = 1$ whenever $a_{ij} \in \mathbb{R}_{>0}$. We apply the same rule to the parameters k_i, l_{1i}, l_{2i}, and s_i.

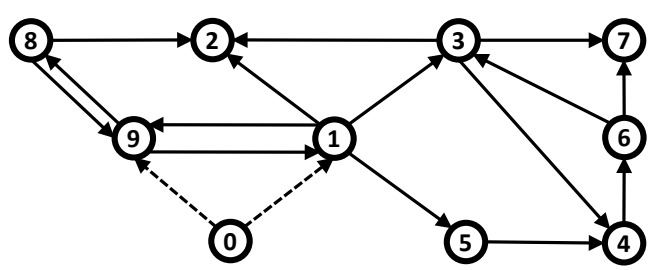

Figure 4.1: The directed communication graph $\mathcal{G}$, which does not include the leader node v_0, and the augmented directed communication graph $\bar{\mathcal{G}}$, which includes the leader node and the edges denoted by dashed arrows.

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4.5.1 First Example

To illustrate the results in Section 4.4.1, we consider nine agents that have fourth-order dynamics, where $x_i(t) = [x_{i1}(t), x_{i2}(t), x_{i3}(t), x_{i4}(t)]^{\mathrm{T}}$, with the following system and input matrices

$$A = \begin{bmatrix} -1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0.000625 & -0.0625 & 0.01 & 0 \\ 1 & -0.5 & 0 & -1 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 \\ 2 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \tag{4.32}$$

and output matrices for the agents in $\mathcal{N}_{\mathrm{p}}$

$$C_i = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \ i = 1,2, \quad C_3 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \tag{4.33}$$

over the graph $\mathcal{G}$ in Figure 4.1. Moreover, the agent-specific dynamical systems of the agents in $\mathcal{N}_{\mathrm{p}}$ are determined by

$$\Gamma_i = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad F_i = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \ i = 1,2, \tag{4.34}$$

$$\Gamma_3 = \begin{bmatrix} -1.5 & 0.5 & -1.5 & 1 \\ -2.25 & 1.25 & -0.25 & 2.25 \\ 1.125 & -0.625 & 0.625 & -0.375 \\ 0.125 & -0.625 & -1.375 & -0.375 \end{bmatrix}, \ F_3 = C_3. \tag{4.35}$$

With the given matrices, Assumptions 4.2.1 and 4.3.1 hold. Assumption 4.3.2 also holds by Theorem 1.9 in [25] since Assumption 1.4 in [25] is satisfied for each agent in $\mathcal{N}_{\mathrm{p}}$.

We take $Q = I_4$ and $R = I_2$ in (4.2). Then, the feedback gain K, given in Lemma 4.2.1, is computed through the unique positive definite solution to (4.2) (e.g., see MATLAB function *lqr*). The coupling gain c is also set to 1. Hence, the conditions of Theorem 1 are satisfied. Note that the regulator equations in (4.20) and (4.21) can be transformed into systems of linear equations (see the proof of Theorem 1.9 in [25]). For the agents in $\mathcal{N}_{\mathrm{p}}$, the resulting systems of linear equations have unique solutions since each matrix of

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coefficients for these systems is nonsingular. With these solutions, the feedforward gains H_i of the agents in $\mathcal{N}_{\mathrm{p}}$ for the first and second distributed controllers, given in Theorems 4.4.1 and 4.4.2, are computed. Thus, our numerical setup satisfies every condition in Theorems 4.4.1 and 4.4.2.

The initial conditions of the agents and their corresponding reference models are given by $x_{i0} = [\frac{i}{4}(-1)^i, 0, \frac{i}{4}(-1)^i, 0]^{\mathrm{T}}$ and $x_{\mathrm{r}i0} = \frac{i}{10}\mathbf{1}_4$ for all $i \in \mathcal{N}$. Moreover, the initial conditions of (4.9) are given by $\delta_{10} = [2, 0, 0.02]^{\mathrm{T}}$, $\delta_{20} = [-2, 0, -0.02]^{\mathrm{T}}$, and $\delta_{30} = \frac{1}{2}\mathbf{1}_4$.

Finally, let $\theta_1(t) = \theta_2(t) = 1$ if $t \in [10, 50)$ sec, $\theta_1(t) = \theta_2(t) = 0$ otherwise. Let $\theta_3(t) = 1$ if $t \in [30, 70)$ sec, $\theta_3(t) = 0$ otherwise. These inform the first three agents that the first and second ones have local objectives only from 10 to 50 sec, and the third one has local objective only from 30 to 70 sec. Now, the responses of all agents with the first and second distributed controllers are illustrated in Figures 4.2 and 4.3. As expected by Theorems 4.4.1 and 4.4.2, every agent in $\mathcal{N}_{\mathrm{p}'}$ obeys the global objective $x_{\mathrm{r}}^{\star}(t)$ of the multiagent system and every agent in $\mathcal{N}_{\mathrm{p}}$ performs its own local objective $\omega_i(t)$ around $y_i^{\star}(t)$ if it is assigned and obeys the global objective otherwise.

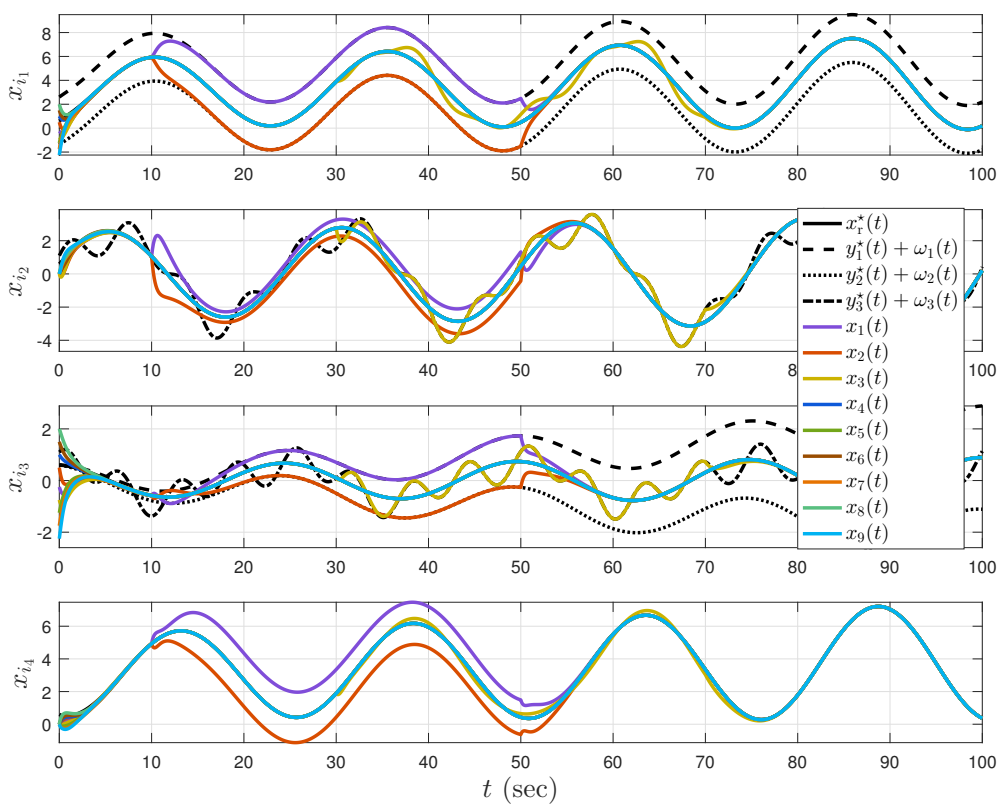

Figure 4.2: Responses of all agents with the first distributed controllers.

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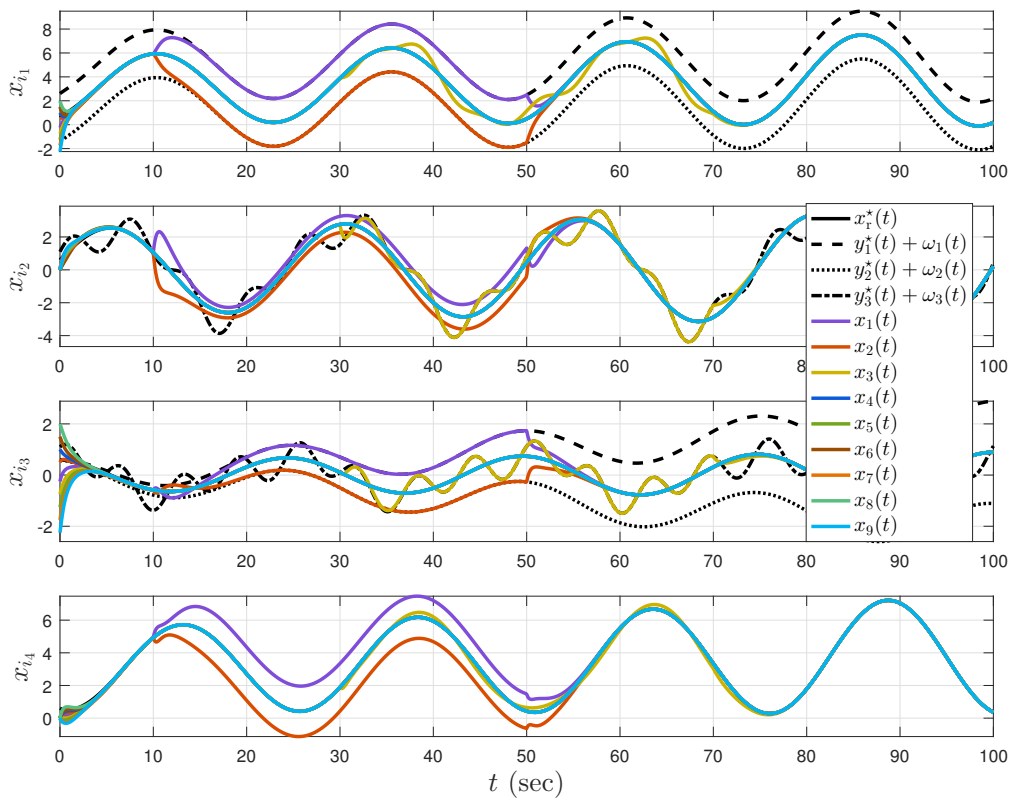

Figure 4.3: Responses of all agents with the second distributed controllers.

4.5.2 Second Example

Now, our aim is to illustrate the results in Section 4.4.2. For brevity, we only give the changes with respect to the previous subsection. First, replace (4.32) by

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \tag{4.36}$$

and (4.33) by

$$C_i = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \; i = 1, 2, 3. \tag{4.37}$$

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Consider the augmented graph $\bar{\mathcal{G}}$ in Figure 4.1. Instead of (4.34) and (4.35), we have $\Gamma_1 = 0$, $\Gamma_2 = A$, $F_1 = F_3 = I_2$, $F_2 = C_1$, and

$$\Gamma_3 = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}. \tag{4.38}$$

Then, we take $Q = \mathrm{diag}(10,1,10,1)$ and $R = I_2$ in (4.2) and the coupling gain $c = 2$. By following the procedure in Section 4.5.1, the feedback gain K and the feedforward gains H_i are obtained. Moreover, all initial conditions are selected as follows: $x_{i0} = [\frac{i}{4}(-1)^i, 0, 0, 0]^{\mathrm{T}}$ and $x_{ri0} = x_{i0}$ for all $i \in \mathcal{N}$; $r_0 = [0,-2,1,0]^{\mathrm{T}}$, $\delta_{10} = [3,2]^{\mathrm{T}}$, $\delta_{20} = [0,2,-1,0]^{\mathrm{T}}$, and $\delta_{30} = [0,0.5]^{\mathrm{T}}$. Let $\theta_1(t) = \theta_2(t) = 1$ if $t \in [15,45)$ sec, $\theta_1(t) = \theta_2(t) = 0$ otherwise. Let $\theta_3(t) = 1$ if $t \in [30,45)$ sec, $\theta_3(t) = 0$ otherwise.

Each agent in this example may be regarded as an undamped vehicle. In particular, the first and third state variables of the agents correspond to the positions in the x and y directions, respectively, while the second and fourth ones correspond to the velocities in the x and y directions, respectively. Because of the output matrices in (4.37), local objectives of the agents in $\mathcal{N}_{\mathrm{p}}$ are related to their positions. In Figures 4.4 and 4.5, the position trajectory of each agent is presented with the first and second distributed controllers, where $y_i = C_i x_i$ with $C_i = C_1$ for all $i \in \mathcal{N}_{\mathrm{p}'}$ and $y^\star = C_1 r$. Also, "o" denotes the position of each agent at $t = 0$, $t = 15$, $t = 30$, and $t = 45$ sec in the subplots. It is observed that every agent in $\mathcal{N}_{\mathrm{p}'}$ synchronizes to the leader in (4.5) whose position trajectory is an ellipse. On the other hand, the agents 1, 2, and 3 track another ellipse, go to the origin, and track another geometric trajectory, respectively, when they are assigned the local tasks and synchronize to the leader otherwise.

4.6 Conclusion

The current literature of distributed control provides useful methods to achieve a wide range of global objectives for multiagent systems. In this paper, we have considered networks of agents with not only global but also local objectives. For linear time-invariant multiagent systems over fixed directed communication graph topologies, we defined agent-specific dynamics to assign local tasks to a subset of agents around the global objective of the multiagent system, constructed reference models for all agents by means of two existing synchronization results, and introduced two classes of distributed controllers. We then system-theoretically stated the considered control problems and solved them by both employing

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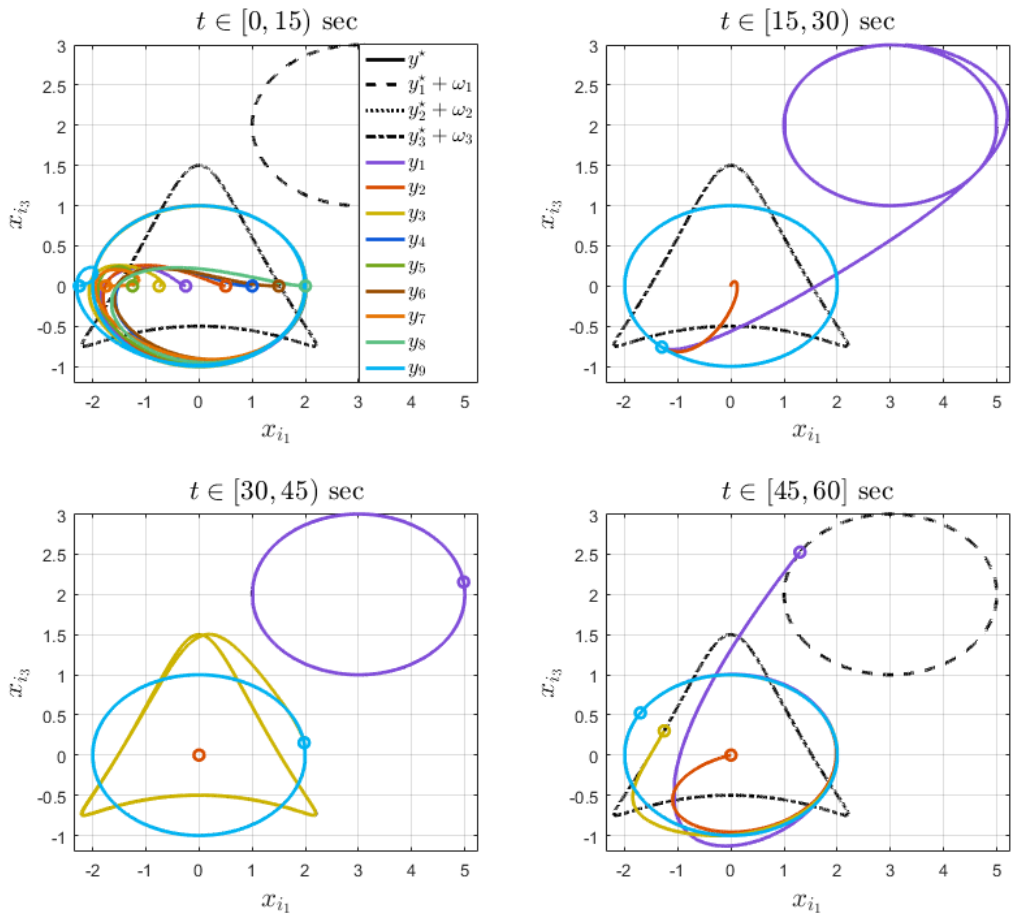

Figure 4.4: Position trajectories of all agents with the first distributed controllers.

the converging-input converging-state property for a class of linear systems and applying the feedforward design methodology from the linear output regulation theory. In short, the second controllers are superior in terms of restriction and dependence on communication graphs (see Remarks 4.4.1 and 4.4.3) and transient performance (see Remark 4.4.4), whereas the first controllers may incur less measurement cost than the second controllers (see Remark 4.3.1).

Although leaderless synchronization and synchronization to a leader are considered in this paper as global objectives, we expect that our framework can be useful for other global objectives such as containment and formation once reference models for agents are constructed through some results addressing these global objectives. Furthermore, it can be desirable in practical applications that some agents perform different local objectives. Our results can also be extended to this case by introducing various agent-specific dynamical systems for each of these agents and utilizing the state of one of the introduced systems together with its

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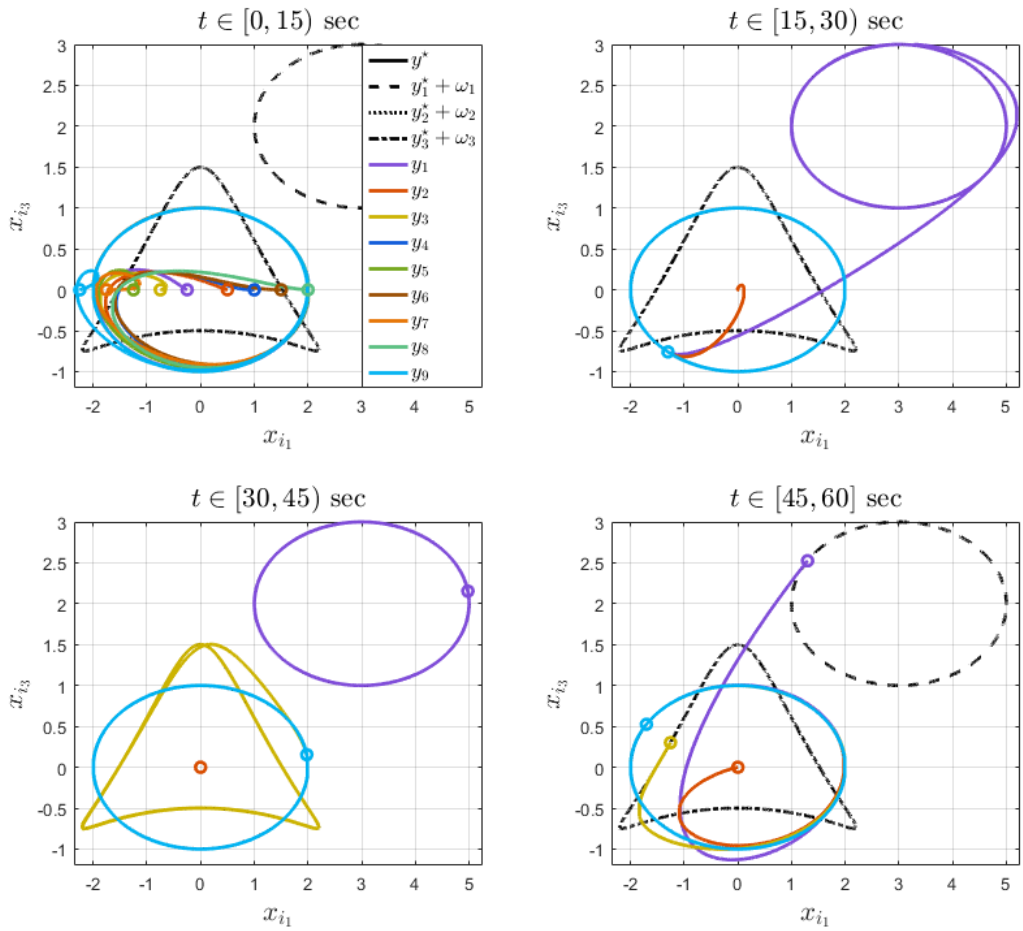

Figure 4.5: Position trajectories of all agents with the second distributed controllers.

corresponding feedforward gain in distributed controllers depending on the current local objective for each of them. Other research directions include, but are not limited to, the accomplishment of agents' local objectives in finite time and the investigation of another problem led by local objectives that are determined directly (i.e., they are not relative to the global one).

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Chapter 5: Concluding Remarks

This work has investigated three aspects of distributed control problems arising from networks of heterogeneous agents or the heterogeneous nature of multiagent systems. In this chapter, we summarize our results and provide a few directions for future research inspired by them.

5.1 Conclusions

We studied the cooperative output regulation problem of heterogeneous linear time-invariant multiagent systems over fixed directed communication graph topologies in Chapter 2. For three internal model-based distributed control laws, namely dynamic state feedback, dynamic output feedback with local measurement, and dynamic output feedback, the solvability of the problem was investigated in two steps. First, the overall closed-loop stability (i.e., global property), which requires both the dynamics of every agent and the communication graph, was assumed and it was proved, under mild assumptions, that the problem is solved. Second, an agent-wise local sufficient condition, which paves the way for independent controller design for each agent, was provided to ensure the global property under standard assumptions. We also reported and addressed a considerable number of gaps in the existing related literature.

In Chapter 3, the definition of the linear cooperative output regulation problem was updated to allow not only the primary synchronization role but also a secondary synchronization role for a distributed dynamic state feedback control law that does not rely on the exchange of its state variables. Similar to Chapter 2, the solvability of the updated problem was investigated by employing the internal model approach and a small-gain theorem. It should be emphasized that the proposed distributed control laws in Chapters 2 and 3 solves the linear robust cooperative output regulation problem as well.

In this dissertation, we also focused on distributed control problems of linear time-invariant multiagent systems with both global and local objectives. Specifically, we defined agent-specific dynamics to assign local tasks to a subset of agents around the global objective of the multiagent system, constructed reference models for all agents by means of two existing synchronization results, and introduced two classes

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of distributed controllers. We then system-theoretically stated the considered control problems and solved them by making use of the converging-input converging-state property for a class of linear systems and the feedforward approach of the linear output regulation theory.

5.2 Recommendations for Future Research

The communication graph was assumed to be fixed throughout this work. However, in practice, it can be switching due to unreliable communication links. Thus, extensions of our results, particularly cooperative output regulation results, to switching topologies would be a major development.

The agent-wise local sufficient conditions in Chapters 2 and 3 are the salient outcomes of this work. For all their importance, the agent-wise local sufficient condition in Chapter 3 can be more conservative than the corresponding one in Chapter 2 because of the following reasons: First, unlike $\mathcal{FA}$, the spectral radius of $\mathcal{Q}$ may not be less than 1. Second, instead of one $\mathcal{L}_2$ gain, two $\mathcal{L}_2$ gains for each agent need to be minimized in Chapter 3. Therefore, further research on relaxation of the agent-wise local sufficient condition for the linear cooperative output regulation problem in Chapter 3 is recommended. In addition, future research can extend the results of Chapter 3 to multiple secondary synchronization roles. As in Chapter 2, dynamic output feedback control laws can also be considered for the problem in Chapter 3.

According to the problem definitions in Chapter 4, agents are expected to achieve their local objectives asymptotically. However, the accomplishment of agents' local objectives in finite time or prescribed time would be desired for time critical applications.

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Appendix A: Solvability of (2.26) and (2.27)

Section III in [1] also studies the solvability of the matrix equations in (2.26) and (2.27), which correspond to the matrix equations given by (6) in [1], with an alternative approach. Specifically, the last paragraph of Section III in [1] lists three sufficient conditions based on Remark 3.8 of [70] to guarantee that these matrix equations have a unique solution. However, it cannot be guaranteed as it is claimed in [1]. This section aims to present the gaps between the conditions and the existence of a unique solution to the matrix equations, propose appropriate modifications that fill these gaps, and explain the motivation behind our approach. For this purpose, we first focus on Definition 3.7 and Remark 3.8 in [70] to fix a problem in [70]. Then, we revisit the conditions listed in [1] to point out the missing one. Finally, a motivational example is provided and the difference between the approach in [1] and the one in this paper is highlighted.

In this paragraph, the notation and the terminology in [70] are adopted and readers are referred to (3.5), (3.6), (3.8), Definition 3.7, and Remark 3.8 in [70]. The problem in [70] is that the conditions of Remark 3.8 do not ensure the stabilizability of the pair given by (3.8). Moreover, this problem is directly transferred to [1]. To illustrate this point, we consider the following system, input, output, and direct feedthrough matrices of the plant; and system matrix of the exosystem

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \; C = \begin{bmatrix} 0.5 & -0.5 \end{bmatrix}, \; D = 0, \; A_1 = 0.$$

It can be easily checked that the plant and the exosystem above satisfy the first and the second conditions of Remark 3.8. Note that $m(s) = s$ is the minimal polynomial of A_1. Then, choose the pair (β_1, σ_1) in (3.6) as follows

$$\beta_1 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \; \sigma_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

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It is obvious that the pair (β_1, σ_1) is controllable and the minimal polynomial of A_1 divides the characteristic polynomial of β_1. Thus, the pair $(G_1, G_2) \triangleq (\beta_1, \sigma_1)$ incorporates a 1-copy internal model of A_1 according to Definition 3.7. Let us now investigate the stabilizability of the pair in (3.8). This pair is not controllable by the controllability matrix test (e.g., see Theorem 12.1 in [59]) and the eigenvalues of the first matrix of this pair are -1, 0, 1, and 2. The eigenvector test for stabilizability (e.g., see Theorem 14.1 in [59]) reveals that unstable eigenvalue 1 is the uncontrollable mode; that is, the pair in (3.8) is not stabilizable. Hence, there do not exist K_1 and K_2 such that A_c defined in (3.5) is Hurwitz. This counterexample to Remark 3.8 is obtained due to the fact that the constructed G_1 violates Property 1.5 in [25]. In fact, J. Huang (personal communication, June 9, 2018) recognizes the problem in Remark 3.8; hence, he adds Property 1.5 as a condition to Lemma 1.26[23] of [25].

In this paper, Definition 2.2.1 modifies the second property of Definition 1.22 given after (1.58) in [25]. This modification guarantees that Property 1.5 in [25] automatically holds if Assumption 2.3.5 holds. Based on the foregoing discussions, it is clear that Remark 2.4.4 is true.

The following two paragraphs adopt the notation and the terminology from [1]. Readers are referred to (5), (6), (7), (8), (10), Definition 2, Lemma 2, Section II.B, and Section III in [1]. It is shown in Section III that if the matrix equations in (8) have solutions X_{1i} and X_{2i} for $i = 1, \ldots, N$, then the ones in (7) have solutions $X_1 = \mathrm{diag}(X_{11}, \ldots, X_{1N})$ and $X_2 = \mathrm{diag}(X_{21}, \ldots, X_{2N})$; that is, the matrix equations in (6) have a solution $X = [X_1^{\mathrm{T}}\ X_2^{\mathrm{T}}]^{\mathrm{T}}$. Furthermore, it is claimed that if the three conditions[24] listed in the last paragraph of Section III hold, then the matrix equations in (8) have unique solutions X_{1i} and X_{2i} for $i = 1, \ldots, N$. However, these conditions do not guarantee the unique solutions. For, consider $A_1 = 0$, $B_1 = 1$, $C_1 = 1$, $D_1 = 0$, $S = 0$, $R = 1$, $P_1 = 1$, $F_1 = 0$, and $G_1 = 1$. It can be easily checked that the listed conditions are satisfied and Property 1.5 in [25] is not violated. Choose $K_1 = 0$ and $H_1 = 0$. From the first matrix equation in (8), we get $1 = 0$, which is a contradiction. We now point out the problem in the claim. First, observe that the matrix equations in (8) can be equivalently written as the matrix equations given by (1.70) and (1.71) in [25]. Then, by Lemma 1.27 in [25], one can note that the following condition is missed in the claim: *$\tilde{A}_i$ given after (10) is Hurwitz*[25] *for $i = 1, \ldots, N$*. It can be shown that this condition, together with the assumption on S, ensures that zero matrices are the *unique* solutions to the off-block-diagonal matrix

[23] We also note that the proof of Lemma 1.26 in [25] is still valid even if Assumption 1.1 in [25] is removed from the hypotheses of Lemma 1.26.

[24] In Section II.B, S is assumed to have no strictly stable modes.

[25] After the suggested modification above, K_i and H_i can always be chosen such that $\tilde{A}_i$ is Hurwitz under the listed conditions.

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equations in (7) by adding $G_c\big((C_c+D_cK_c)X_1+D_cH_cX_2-R_c\big)$ to the left side of the second equation in (7) that gives an equivalent form of (7) and applying the first part of Proposition A.2 in [25]. In conclusion, if the assumption on S holds, the third condition in the list holds for $i=1,\ldots,N$, and $\tilde{A}_i$ *is Hurwitz for* $i=1,\ldots,N$, then the matrix equations in (6) have a *unique* solution X.

According to Lemma 2, the problem in Definition 2 is solved if the assumption on S holds, A_l given after (5) is Hurwitz, and the matrix equations in (6) have a unique solution X. Although the approach utilized during the derivation of the listed conditions does not take into account the assumption on A_l, one may wonder the answer of the following question: *Let the listed conditions hold and A_l be Hurwitz. Then, can we conclude that $\tilde{A}_i$ is Hurwitz for $i=1,\ldots,N$?* The answer is *no*. That is, the missing condition cannot be satisfied by assuming that the listed conditions hold and A_l is Hurwitz. To clarify this point, consider the system parameters of the agents, the system matrix of the exosystem, and the adjacency matrix of $\mathcal{G}^*$

$$
\begin{aligned}
A_1 &= \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix},\ B_1=\begin{bmatrix} 1 & 0.5 \\ 0 & 0.25 \end{bmatrix},\ C_1=\begin{bmatrix} 1 & -0.5 \end{bmatrix},\ D_1=0,\\
A_2 &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},\ B_2=\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},\ C_2=\begin{bmatrix} 1 & 0 & 0 \end{bmatrix},\ D_2=0,\\
A_3 &= 1,\ B_3=-1,\ C_3=1,\ D_3=0,\ S=0,\\
Q^* &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.5 & 0 & 0 & 0.5 \\ 0 & 0.5 & 0 & 0.5 \\ 0 & 0.5 & 0.5 & 0 \end{bmatrix}.
\end{aligned}
$$

Choose $(F_i,G_i)=(0,1)$, $i=1,2,3$. It can be easily checked that the listed conditions are satisfied and Property 1.5 in [25] is not violated. One can also obtain W, which is required to construct A_l, from Q^*. Then, choose the remaining parameters of the controllers as follows

$$
\begin{aligned}
K_1 &= \begin{bmatrix} 2.6752 & 9.6624 \\ -10.6752 & -24.6624 \end{bmatrix},\ H_1=\begin{bmatrix} -6.4 \\ 6.4 \end{bmatrix},\\
K_2 &= -\begin{bmatrix} 104.56 & 57.936 & 14.828 \end{bmatrix},\ H_2=-80,\ K_3=0.8,\ H_3=1.
\end{aligned}
$$

With this setup, it can be verified that $\tilde{A}_3$ is not Hurwitz even though A_l is Hurwitz.

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Based on the previous example, the following question arises: *Is the missing condition in [1] necessary to ensure that the matrix equations given by (6) in [1] have a unique solution?* In fact, this question is the motivation behind the key lemma (i.e., Lemma 2.4.3) of this paper and the answer is *no*. In contrast to Section III in [1], the approach in Lemma 2.4.3 does not decompose matrix equations, which consist of the overall dynamics of the multiagent system, into matrix equations, which deal with the dynamics of each agent separately; hence, the missing condition in [1] is not required in Lemma 2.4.3. Furthermore, not only dynamic state feedback but also dynamic output feedback with local measurement and dynamic output feedback effectively utilize Lemma 2.4.3 to solve the stated problem in Definition 2.3.1 (see Theorems 2.4.1, 2.4.3, and 2.4.4).

Since the proof of Theorem 1 and the statement of Theorem 4 in [39] use the approach in Section III of [1], we believe that the discussion in this section will also be helpful for the readers of the results in [39].

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Appendix B: On Theorem 2 in [1]

In this section, the notation and the terminology in [1] are adopted and readers are referred to (5), (10), (15), and Theorem 2 in [1]. Now, consider the system parameters of the agent, the system matrix of the exosystem, and the adjacency matrix of $\mathcal{G}^*$ given by

$$A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, B_1 = I_3, C_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, D_1 = 0, S = 0, Q^* = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}.$$

Choose $(F_1, G_1) = (0, I_2)$ and

$$K_1 = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 2 \end{bmatrix}, H_1 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ 0 & 0 \end{bmatrix}.$$

Note that $W = 1$ from Q^*; hence, A_l given after (5) is nothing but $\tilde{A}_1$ given after (10). With this setup, one can verify that $T_1(s)$ given before Theorem 2 is stable and the condition in (15) is automatically satisfied, but A_l is not Hurwitz. This counterexample is obtained because the realization of $T_1(s)$ is neither stabilizable nor detectable. In fact, a loss of one of them is enough to find a counterexample.

The above setup also applies to Theorem 5 in [39] since it relies on Theorem 2 and its conditions are satisfied. It should be noted that although Assumptions 1-4 in [39] and Property 1.5 in [25] are not listed in the hypotheses of Theorem 5 in [39], this counterexample does not violate them.

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Appendix C: Proofs of Lemmas 4.2.3-4.2.5 and 4.3.2

Proof of Lemma 4.2.3. Since $\lim_{t\to\infty} f(t) = h \in \mathbb{R}^n$, there exists a $T \in \mathbb{R}_{>0}$ such that $\|f(t)-h\|_2 < 1, \forall t > T$. Hence, the triangle inequality $\|f(t)\|_2 \leq \|f(t)-h\|_2 + \|h\|_2$ yields $\|f(t)\|_2 < \|h\|_2 + 1, \forall t > T$, so f is bounded in (T,∞). Since $[0,T]$ is compact and f is piecewise continuous on $[0,T]$, f is bounded on $[0,T]$ by Proposition 2.18 in [66]. Thus, the result follows. ■

Proof of Lemma 4.2.4. Since $\lim_{t\to\infty} g(t) = 0$ and the range of g is a subset of $\{0,1\}$, there exists a $T \in \mathbb{R}_{>0}$ such that $g(t) = 0, \forall t > T$. This implies that $(gf)(t) = 0, \forall t > T$ and completes the proof of the lemma. ■

Proof of Lemma 4.2.5. Since the system in (4.7) is linear time-invariant and A_{c} is Hurwitz, (4.7) is input-to-state stable (e.g., see the fourth part of Exercise 7.3.11 in [71], Chapter 4.9 in [55], and Chapter 4.5 in [66]). Since (4.7) is input-to-state stable and $\eta(t)$ is piecewise continuous, $\lim_{t\to\infty} \eta(t) = 0$ implies that $\lim_{t\to\infty} \zeta(t) = 0$ for all $\zeta_0 \in \mathbb{R}^n$ (e.g., see the second part of Exercise 7.3.11 in [71], Exercise 4.58 in [55], and the proof[26] given after Definition 4.6 in [66]). ■

Proof of Lemma 4.3.2. Under the hypotheses of Lemma 4.3.1, (4.22) and (4.23) have a unique solution $\tilde{X}_i$. Let $\tilde{\zeta}_{i0} \in \mathbb{R}^n$ and $\delta_{i0} \in \mathbb{R}^{h_i}$ be given. Define $\bar{\zeta}_i(t) \triangleq \tilde{\zeta}_i(t) - \tilde{X}_i\delta_i(t)$. Then, using (4.9), (4.10) and (4.22)-(4.25), we have

$$\dot{\bar{\zeta}}_i(t) = (A+BK_{1i})\bar{\zeta}_i(t) + \phi_i(t), \quad \bar{\zeta}_i(0) = \bar{\zeta}_{i0}, \quad t \geq 0, \tag{A.1}$$

$$\tilde{\beta}_i(t) = C_i\bar{\zeta}_i(t). \tag{A.2}$$

Since $A+BK_{1i}$ is Hurwitz and $\phi_i(t)$ is a convergent piecewise continuous function of t to 0, $\lim_{t\to\infty} \bar{\zeta}_i(t) = 0$ by Lemma 4.2.5. It now follows from (A.2) that $\lim_{t\to\infty} \tilde{\beta}_i(t) = 0$. ■

[26] Although it is assumed in [66] that $\eta(t)$ is continuous, bounded, and $\lim_{t\to\infty} \eta(t) = 0$, the proof in [66] is still valid when $\eta(t)$ is piecewise continuous and $\lim_{t\to\infty} \eta(t) = 0$ since the boundedness of $\eta(t)$ immediately follows from Lemma 4.2.3.

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Appendix D: Errata

i) Page 10, Line 18: Change "root node" to "node".

ii) Page 11, Line 6: Due to the given condition $b)$ of Definition 2.2.1, $s_l = s_1$ for $l = 2, \ldots, p$.

iii) Page 20, Remark 2.4.3 : For clarity, "$\lim_{t\to\infty} A_0\omega(t) - \dot{\omega}(t) = A_0 \lim_{t\to\infty} \omega(t) - e^{A_0 t}\omega_0 + \lim_{t\to\infty} A_0 e^{A_0 t}\omega_0 - \dot{\omega}(t) = 0$" should read "$\lim_{t\to\infty}(A_0\omega(t) - \dot{\omega}(t)) = A_0\big(\lim_{t\to\infty}(\omega(t) - e^{A_0 t}\omega_0)\big) + \lim_{t\to\infty}(A_0 e^{A_0 t}\omega_0 - \dot{\omega}(t)) = 0$".

iv) Page 34, Section 3.1.1: See Section 1.2 for better related literature.

v) Page 37, Line 10: Change "root node" to "node".

vi) Page 37, Last Line: Due to the given condition $ii)$ of Definition 3.2.1, $h_l = h_1$ for $l = 2, \ldots, p$.

vii) Page 40, Line 6: Change "Generalizing the definition of the linear cooperative output regulation problem in [34], the problem ..." to "The problem ...".

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